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Which symmetry? Noether, Weyl, and conservation of electric charge

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Abstract

In 1918, Emmy Noether published a (now famous) theorem establishing a general connection between continuous 'global' symmetries and conserved quantities. In fact, Noether's paper contains two theorems, and the second of these deals with 'local' symmetries; *prima facie*, this second theorem has nothing to do with conserved quantities. In the same year, Hermann Weyl independently made the first attempt to derive conservation of electric charge from a postulated gauge symmetry. In the light of Noether's work, it is puzzling that Weyl's argument uses *local* gauge symmetry. This paper explores the relationships between Weyl's work, Noether's two theorems, and the modern connection between gauge symmetry and conservation of electric charge. This includes showing that Weyl's connection is essentially an application of Noether's second theorem, with a novel twist.

Keywords: Noether's theorem; Gauge symmetry; Charge conservation; Local symmetry

1. Introduction

The idea of connecting conservation of electric charge with gauge symmetry goes back to 1918 and to Hermann Weyl's attempt to produce a unified theory of electromagnetism and gravitation by generalising the geometry on which General Relativity is based (Weyl, 1918a; see also Weyl, 1918b, 3rd edition). It is well-known that this attempted unification failed, and that Weyl re-applied the gauge idea in the context of quantum theory in 1929,[1] there giving us his 'gauge principle' which has

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[1] The term 'gauge' originates from the translation of Weyl's work into English. A better translation of the original idea might perhaps have been 'scale'. The use of the term 'gauge' in quantum theory has nothing to do with scale, of course, and is just an accident of history.

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been so powerful in the latter half of this century.[2] According to the standard account, Weyl's claim to have connected conservation of electric charge with gauge symmetry comes to fruition in relativistic field theory.

The question addressed in this paper springs from the following observation. In his 1918 theory, Weyl introduced local gauge transformations (transformations that depend on arbitrary functions of space and time), and it is *local* gauge symmetry that he connects with conservation of electric charge. According to the standard modern account, however, *global* gauge symmetry is invoked to deliver conservat ion of electric charge (see, for example, Leader & Predazzi, 1996; Ryder, 1985; Sakurai, 1964; Schweber, 1961; Sterman, 1993; Weinberg, 1995). Which is the correct symmetry to connect with charge conservation? This question might seem straightforward on the surface, but it turns out that a rather interesting story lies behind any satisfactory answer. The story involves a triangle of relationships, none of which has been adequately addressed in the literature. The relationships are between Weyl's work, relativistic field theory, and Noether's theorems.

In the same year that Weyl published his original paper connecting conservation of electric charge with gauge symmetry, Emmy Noether published a paper (Noether, 1918) that is now famous for 'Noether's theorem'. This theorem makes a general connection between conserved quantities and continuous symmetry transformations that depend on constant parameters (for example, a spatial translation in the x-direction, $x \rightarrow x' = x + a$, where a is a constant); such a transformation is a global transformation. This theorem is in fact the first of two theorems proved in the 1918 paper, both of which are derived from a variational problem. The second theorem is less well-known, and applies to symmetry transformations that depend on arbitrary functions of space and time, and on their derivatives; such a transformation is a local transformation.

The first theorem works straightforwardly for the continuous symmetries of space and time in classical mechanics, giving conservation of linear and angular momentum, energy, and so forth. However, when we come to gauge symmetry and conservation of electric charge, things are not so straightforward, since both theorems come into play. The first theorem applies to global gauge symmetry and the second theorem applies to local gauge symmetry. In modern relativistic field theory, the standard account connects conservation of electric charge with gauge symmetry via Noether's first theorem. Any connection using Noether's first theorem must come by applying it to the global subgroup of the local gauge group. This leaves us with the following questions: what is the role of the second theorem in locally gauge invariant theories, and what is the relationship between the second theorem and the first theorem (as applied to the global subgroup) in such theories? Although Noether's first theorem has received thorough treatment (see especially Hill, 1951; Doughty, 1990), very little attention has been given to her second theorem (and to cases where both

[2] Weyl (1929). For a discussion of this 1929 paper, and of the work which preceded it by Schrödinger, London and Fock, see O'Raifeartaigh (1997).

theorems apply). Clarifying the roles of the first and second Noether theorems in relation to modern relativistic field theory gives us one arm of the triangular relationship.

The modern-day connection goes via global gauge symmetry, but in 1918 Weyl claimed to have connected conservation of electric charge to local gauge symmetry. Despite perennial interest in Weyl's 1918 work, and the recent revival in interest in the early history of gauge theory (see especially O'Raifeartaigh, 1997), the relationship, if any, between Weyl's 1918 work and Noether's second theorem has never been made clear. This is the second arm of the three-way relationship, and we will clarify it here.

The final arm that will be addressed is between Weyl's 1918 work and modern relativistic field theory, where Weyl's 1918 work is usually said to come to fruition. Again, despite the interest in Weyl's 1918 work, the relationship between (a) Weyl's 1918 connection between local gauge invariance and conservation of electric charge, and (b) the modern connection between global gauge invariance and conservation of electric charge, has never been clarified. That it stands in need of clarification is emphasised by the fact that the latter proceeds via Noether's first theorem, and the former does not.

So, the question before us is: which symmetry is the correct symmetry to associate with conservation of electric charge—global gauge symmetry or local gauge symmetry? In order to address this question, we begin by stating Noether's two theorems (Section 2). We will then see how a careful understanding of Noether's two theorems bears on the case of electromagnetism and conservation of electric charge. In Section 3 we discuss the modern textbook derivation, and in Section 4 we examine the relationship between Weyl's work and Noether's theorems. Section 5 discusses the relationship between Weyl's work and relativistic field theory, and Section 6 tackles the application of Noether's second theorem in relativistic field theory. Section 7 takes a brief look at a distinction made by Noether between 'proper' and 'improper' conservation laws, and Section 8 draws all these strands together to address the question 'Which Symmetry?'.

2. Noether's two theorems

Noether's second theorem is central to the story that will be told in this paper. As already mentioned, this theorem has received much less attention than the first theorem. For example, in *Emmy Noether: A Tribute to Her Life and Work* (Brewer and Smith, 1981), McShane's chapter on the calculus of variations discusses the first theorem in detail but merely states the second theorem without proof or discussion. In the introduction to Noether's *Collected Papers* (Jacobson, 1983), the commentary on the 1918 paper consists of an extensive quote from Feza Gursey, with no mention of the second theorem. There is also very little on the second theorem in the standard history of field theory and gauge theory literature: there is no discussion in O'Raifeartaigh (1997), Vizgin (1994), Moriyasu (1982) or Hill (1951), for example;

and, although the second theorem is cited in Kastrup's excellent 1987 paper, it is not discussed in any detail. There is, however, a recent paper by Byers (1999) that discusses the relationship between the second theorem and General Relativity, and an excellent paper by Rowe (1999) that discusses the historical context in which Noether came to derive her two theorems (see also Pais, 1987; Sauer, 1999). If we turn to the physics textbooks, Lanczos (1970) and Doughty (1990) are seminal texts on variational techniques, Doughty especially giving a detailed and thorough derivation of Noether's first theorem using variational principles, but neither discusses the second theorem. In standard field theory textbooks (such as those cited in the introduction), Noether's first theorem is widely cited, but the second does not appear. The key references for the second theorem in the physics literature are Utiyama's (1959) follow-up paper to his 1956 paper on local gauge theories, where he recognises the connection to Noether's work,[3] and the excellent paper by Trautman (1962) that discusses the second theorem in the context of General Relativity. Reviving a more widespread interest in Noether's second theorem is of more than historical interest: failure to appreciate the domains of applicability of these two distinct theorems has given rise to ongoing mistakes and confusion in the physics literature (see Section 3 and the appendix below for two examples), and current discussions on conservation of energy in General Relativity would benefit from a thorough general understanding of the two theorems. Finally, Noether's paper is difficult to get hold of in English translation. For these reasons, and because the substance of this paper requires an accurate reading of Noether's 1918 paper, I begin by presenting the content of Noether's two theorems as given by Noether herself.

Noether's theorems apply to Lagrangians and Lagrangian densities depending on an arbitrary number of fields with arbitrary numbers of derivatives. However, we will simplify our discussion to consider Lagrangian densities, L, depending on ψ_i, $\partial_\mu\psi_i$, and x^μ, and no higher derivatives of ψ_i, since this is all that we will need for the purposes of this paper.[4] The i indexes each field ψ_i on which the Lagrangian depends. Noether derives her theorems by considering the following variational problem, applied to the action $S = \int L\,\mathrm{d}^4x$. We begin by forming the first variation δS, in which we vary both the independent and the dependent variables (x^μ, and ψ_i, $\partial_\mu\psi_i$, respectively, in our case), we include the boundary in the variation, and we discard

[3] Utiyama goes beyond the results proved explicitly by Noether in 1918. See Brading and Brown (2001) where 'Utiyama's theorem' is discussed and where references may be found to current work on the Noether variational problem falling outside the scope of this paper.

[4] Noether's own statement of the two theorems is as follows:

I. If the integral I is invariant with respect to a G_ρ, then ρ linearly independent combinations of the Lagrange expressions become divergences—and from this, conversely, invariance of I with respect to a G will follow. The theorem holds good even in the limiting case of infinitely many parameters.

II. If the integral I is invariant with respect to a $G_{\infty\rho}$ in which the arbitrary functions occur up to the σ-th derivative, then there subsist ρ identity relationships between the Lagrangian expressions and their derivatives up to the σ-th order. In this case also, the converse holds.

(In Noether's terminology, G_ρ is a continuous group depending on constant parameters, and $G_{\infty\rho}$ is a continuous group depending on arbitrary functions and their derivatives.)

the second and higher order contributions to the variation. We then require that the variation is an infinitesimal symmetry transformation, and hence set $\delta S = 0$.[5]

Before we proceed, I need to introduce one piece of terminology and short-hand notation. Noether derives and presents both her theorems in terms of the Euler derivative, or what she calls the 'Lagrange expression':

$$E_i := \frac{\partial L}{\partial \psi_i} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi_i)} \right), \tag{1}$$

which, when set to zero, gives the Euler-Lagrange equations. We will use this terminology in what follows.

In deriving the consequences of the above variational problem, the first step—common to both theorems—is to show that if the action S is invariant under some group of transformations, then

$$\sum_i E_i \delta_0 \psi_i \mathrel{\hat{=}} \sum_i \partial_\mu B_i^\mu, \tag{2}$$

where

1. δx^μ and $\delta\psi_i = \delta_0\psi_i + (\partial_\mu\psi_i)\delta x^\mu$ are the infinitesimal variations in x^μ and ψ_i, respectively, brought about by the symmetry transformation;
2. $\delta_0\psi_i$ is the change in ψ_i at a fixed co-ordinate: $\delta_0\psi_i = \psi_i'(x) - \psi_i(x)$;
3. B_i^μ has the form $(L - \frac{\partial L}{\partial(\partial_\nu\psi_i)}\partial_\nu\psi_i)\delta x^\mu + \frac{\partial L}{\partial(\partial_\mu\psi_i)}\delta\psi_i$.[6]

Note also that here, and throughout this paper, we use the following conventions:

1. the Einstein convention to sum over Greek indices; all other summations are expressed explicitly;
2. the symbol '$\hat{=}$' to indicate those equations that hold independently of whether or not the Euler–Lagrange equations of motion are satisfied.

Theorem 1. *If the action S is invariant under a continuous group of transformations depending smoothly on ρ independent constant parameters ω_k $(k = 1, 2, \ldots, \rho)$,[7] then* (2) *implies the ρ relationships*

$$\sum_i E_i \frac{\partial(\delta_0\psi_i)}{\partial(\delta\omega^k)} \mathrel{\hat{=}} \partial_\mu j_k^\mu, \tag{3}$$

[5] In this context, a symmetry transformation is a transformation that preserves the explicit form of the Euler–Lagrange equations. The connection between this symmetry requirement and $\delta S = 0$ (which is a sufficient condition for preserving the Euler–Lagrange equations) is explained in detail in Doughty (1990, Sections 9.2 and 9.5). It is well-known that this condition may be weakened (see Doughty and especially Trautman, 1962). For further discussion see Brading and Brown (2001).

[6] If the equations of motion are satisfied, the left-hand side of (2) is zero, and we have $\sum_i \partial_\mu B_i^\mu = 0$. This is the expression used for conserved currents by Jackiw et al. (1994). This is undesirable because the δx and $\delta\psi_i$ then appear in the current, something that can be avoided by proceeding to Noether's theorems. See Brading and Brown (2001), for further discussion.

[7] This is a global symmetry group.

where j^μ_k *is the Noether current associated with the parameter* ω^k:

$$j^\mu_k = \sum_i \left(L - \frac{\partial L}{\partial(\partial_\nu \psi_i)} \partial_\nu \psi_i \right) \frac{\partial(\delta x^\mu)}{\partial(\delta\omega_k)} + \frac{\partial L}{\partial(\partial_\mu \psi_i)} \frac{\partial(\delta\psi_i)}{\partial(\delta\omega_k)}. \tag{4}$$

(Here the 'δ' in $\delta\omega_k$ *does not* indicate a variation, but is used to emphasise that we take infinitesimal ω_k. I have used this potentially confusing notation so as to maintain consistency with Doughty (1990) and Weyl (1918a, b).)

From (3), if the equations of motion are satisfied, then there are ρ continuity equations

$$\partial_\mu j^\mu_k = 0, \tag{5}$$

one for every constant parameter ω_k on which the symmetry group depends.[8]

As I noted in the introduction, this first theorem is the one that is at work in the familiar derivations of conservation of linear momentum from spatial translation invariance and so forth. So, in fact what Noether discusses is not conserved quantities Q_a as such, but currents j^μ_k satisfying continuity equations. Textbooks often move straight from the conclusion that j^μ_k satisfies a continuity equation to an associated claim about Q_a being conserved, i.e. from the claim that $\partial_\mu j^\mu_k = 0$ to $(\mathrm{d}/\mathrm{d}t)Q_k = 0$ where $Q_k := \int_R \mathrm{d}^3x j^0_k(x)$. However, this is valid only for appropriate boundary conditions.[9]

Theorem 2. *If the action S is invariant under a continuous group of transformations depending smoothly on* ρ *independent arbitrary functions* $p_k(x)$ $(k = 1, 2, \dots, \rho)$ *and their first derivatives,*[10] *then* (2) *implies the* ρ *relationships*

$$\sum_i E_i a_{ki} \stackrel{\circ}{=} \sum_i \partial_\mu (E_i b^\mu_{ki}), \tag{6}$$

where a_{ki} *and* b^μ_{ki} *are functions of* ψ_i, $\partial_\mu \psi_i$, *and* x^μ, *as defined in what follows.*

Let δp_k and $\partial_\mu(\delta p_k)$ together constitute infinitesimal symmetry transformations (the δ here once again being used to emphasise that we take infinitesimal p_k, and not

[8] To see that j^μ rather than B^μ is the correct Noether current, see Brading and Brown (2001).

[9] The integration process from the conserved current to the conserved charge goes via Gauss' theorem as follows. First we have: $\partial_\mu j^\mu = 0$. Integrating $\partial_\mu j^\mu$ over a 3-volume R, with $t =$ constant,

$$\int_R \mathrm{d}^3x\, \partial_\mu j^\mu = \int_R \mathrm{d}^3x\, \partial_0 j^0 + \int_R \mathrm{d}^3x\, \partial_i j^i = \frac{\mathrm{d}}{\mathrm{d}t}\int_R \mathrm{d}^3x\, j^0 + \oint_S \mathrm{d}S\, \hat{n}_i j^i,$$

where we have used Gauss' theorem to convert a volume integral $\int_R \mathrm{d}^3x\, \partial_i j^i$ into a surface integral $\oint_S \mathrm{d}S\, \hat{n}_i j^i$. Now, if the fields fall off with distance r such that, as r increases, the net outflow from the region enclosed by S falls off more quickly than the volume enclosed by S increases, then for sufficiently large r there will be no net outflow across the surface S, and $\oint \mathrm{d}S\, \hat{n}_i j^i = 0$. This is our boundary condition, and, given the continuity equation, it can be met only if the region enclosed by S contains no sources. Then we have: $0 = \partial_\mu j^\mu = (\mathrm{d}/\mathrm{d}t)\int_R \mathrm{d}^3x\, \hat{n}_0 j^0 = (\mathrm{d}/\mathrm{d}t)Q$.

[10] This is a local symmetry group. The restriction to the first derivative is again imposed for convenience, since this is all we need in what follows. Noether states and proves her results with no such restriction.

to indicate a variation). Then, since $\delta_0\psi_i$ is linear in δp_k, we can write

$$\delta_0\psi_i = \sum_k \{a_{ki}\delta p_k + b^{\mu}_{ki}\partial_\mu(\delta p_k)\}. \tag{7}$$

Thus, Noether's second theorem gives dependencies between the Lagrange expressions (following Noether's terminology) (1), and their derivatives. What this means will become clear when we discuss specific applications of the theorem below.

These are the two theorems as presented by Noether (with our restriction to $L = L(\psi_i, \partial_\mu\psi_i, x^\mu)$). It is the first theorem, and not the second, that is explicitly concerned with conserved quantities.

I turn now to discuss the relationship between the two theorems.

Suppose that the action S is invariant under a continuous group of transformations depending on ρ arbitrary functions $\rho(x)_k$ (a local symmetry group), and that this group admits of a non-trivial global subgroup (where by global subgroup we mean a subgroup of transformations for which $p_k = \text{constant}$). Then the second theorem applies to the local invariance group of S, and the first theorem applies to the global subgroup. The first theorem gives us divergence relations (3) and the second theorem gives us dependencies between the Lagrange expressions and their derivatives (6). Noether discusses this case in Section 6 of her paper, and shows that the divergence relations must be consequences of the dependencies, and in particular linear combinations of the dependencies.[11] She writes (Noether, 1918; translation taken from Tavel, 1971):

> I shall refer to divergence relationships in which the j^μ_k can be composed from the Lagrange expressions and their derivatives in the specified manner as 'improper', and to all others as 'proper'. (p. 202)

Noether then discusses the specific case of energy conservation in General Relativity (see Brading & Brown, 2001). For our purposes, the general lesson is that conservation laws arrived at by applying the first theorem to the global subgroup of local group are 'improper' conservation laws, and we will discuss what this means in Section 7, below, when we consider the specific case of electric charge conservation.

3. Relativistic field theory and Noether's first theorem

The standard textbook presentation of the connection between conservation of electric charge and gauge symmetry in relativistic field theory involves Noether's first theorem. It can be found, to various levels of detail, in most quantum field theory textbooks, such as those referred to in the introduction. All of these books discuss both global and local gauge symmetry, but none mentions Noether's second theorem. For the purposes of simplicity, I begin by focussing on a single presentation by way of example, that of Ryder (1985). I choose Ryder because his is one of the more detailed presentations, and because his text is widely used.

[11] See also Trautman (1962, p. 178) and Brading and Brown (2001).

Ryder begins by applying Noether's first theorem to the Lagrangian associated with the Klein–Gordon equation for a relativistic complex scalar field, L_m:

$$L_m = \partial_\mu \psi \partial^\mu \psi^* - m^2 \psi \psi^*. \tag{8}$$

This Lagrangian is invariant under global phase transformations of the wavefunction, and from this Ryder derives the corresponding Noether current:

$$j^\mu_{L_m} = i(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*). \tag{9}$$

Integrating this to yield a conserved quantity, Ryder (1985, p. 91) writes: "This (real) quantity *we should like* to identify with charge".

What else is needed before we can make this identification? Ryder's next step is to observe that L_m is not invariant under local phase transformations of the wavefunction. That is to say, although $\psi \to \psi' = \psi e^{-i\theta}$ leaves L_m invariant if θ is a constant (i.e. the transformation is global), if $\theta = \theta(x, t)$ (i.e. the transformation is local), then the transformation from ψ to ψ' no longer leaves L_m invariant. In order to create a Lagrangian that remains invariant under a local transformation, we introduce a 'four-vector' A_μ, and we transform A_μ jointly with ψ and ψ^* according to the following rule:

$$\psi \to \psi' = \psi e^{-iq\theta}, \quad \psi^* \to \psi^{*\prime} = \psi^* e^{iq\theta}, \quad A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta, \tag{10}$$

where q, the charge on the electron, is introduced as a coupling constant.[12] Together, transformations (10) constitute a gauge transformation. This enables us to construct a locally gauge invariant Lagrangian. Finally, we add an extra term in A_μ but not in ψ, which is itself locally gauge invariant, giving us our total, locally gauge invariant Lagrangian

$$L_{\text{total}} = D_\mu \psi D^\mu \psi^* - m^2 \psi \psi^* - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{11}$$

where $D_\mu = (\partial_\mu + iqA_\mu)$ is the covariant derivative, and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. From here, there are two main ways in which to proceed. Ryder obtains the Euler–Lagrange equations for A_μ, which are identified as the inhomogeneous Maxwell equations

$$\partial_\nu F^{\mu\nu} = j^\mu, \tag{12}$$

where j^μ has the form[13]

$$j^\mu = iq(\psi^* D^\mu \psi - \psi D^\mu \psi^*). \tag{13}$$

Then, in virtue of the anti-symmetry of $F^{\mu\nu}$, $\partial_\mu \partial_\nu F^{\mu\nu}$ vanishes and Ryder concludes that this modified current j^μ is the conserved current associated with the Lagrangian (1). The other way of proceeding is via Noether's first theorem once again. Notice

[12] For reasons of overall consistency, I differ from Ryder in placing q in the transformation of ψ rather than of A_μ. This choice corresponds to widespread use.

[13] This current differs from Ryder's by a factor of q, due to the placement of q in the gauge transformation of ψ rather than of A_μ (see previous note). The j^μ quoted here is consistent with Maxwell's equations and with widespread usage.

that L_{total} is invariant under global gauge transformations as well as local gauge transformations. This global symmetry is a special case of the local symmetry in which $\theta = \theta(x, t)$ is set to $\theta =$ constant. As a result, the gauge transformation leaves A_μ invariant and only the wavefunction ψ changes. If we apply Noether's first theorem to the global subgroup of the full gauge group of L_{total}, we obtain the modified conserved current (13).[14,15]

One final remark before moving on. Recall that in her paper Noether distinguishes between 'proper' conservation laws and 'improper' conservation laws. In Noether's terminology, therefore, conservation of electric charge in relativistic field theory, derived via the first theorem, is an 'improper' conservation law. I discuss the significance of this in Section 7, below.

4. Weyl and Noether's theorems

The fact that the Lagrangian L_{total} of relativistic field theory is invariant under the full local gauge group means that Noether's second theorem comes into play. Before turning to the application of the second theorem to L_{total}, I want to look at what Weyl was doing, because it sheds light on the application of Noether's second theorem in relativistic field theory. Weyl was clearly claiming to connect conservation of electric charge to *local* gauge invariance, and the question at issue here is what the relationship is between Weyl's work and Noether. This will lead us into Section 5, where we discuss the relationship between Weyl's work and the standard textbook account discussed in Section 3, and into Section 6, where we turn to the relationship between relativistic field theory and Noether's second theorem.

4.1. Weyl's 1918 theory

In his 1918 paper 'Gravitation and Electricity'[16] Weyl set out to provide a unified field theory by generalising the geometry on which General Relativity is based. Weyl sought to impose a 'rigorous locality' by introducing a geometry in which not only the orientation of vectors may be non-integrable (as in

[14] In 1990–91 there was an exchange in *The American Journal of Physics* (Karatas & Kowalski, 1990; Al-Kuwari & Taha, 1991) concerning whether local gauge symmetry adds any new Noether charges to those arising from global gauge symmetry. This exchange deserves a more detailed discussion, but the most important feature of the correct answer is already evident. Noether's first theorem applies only to global symmetries, and the conserved quantities arising in locally gauge invariant theories result from the application of the first theorem to the rigid subgroup of the gauge group (i.e., to the global symmetry). Therefore, in the standard approach, the same symmetry is in play in both cases, and the same Noether charge results.

[15] The case of Maxwell electromagnetism (i.e. electromagnetism without a gauge-dependent matter field) and conservation of electric charge is discussed in the appendix. There, I point out that because there is no non-trivial rigid subgroup in this case, the first theorem cannot be used to derive conservation of electric charge in this way.

[16] The English translation referred to here of Weyl (1918a) is in O'Raifeartaigh (1997).

General Relativity) but also their lengths.[17] Having developed his geometry, Weyl then goes on to discuss its proposed application to physics.[18] He writes (O'Raifeartaigh, 1997):

> We shall show that: just as according to the researches of Hilbert, Lorentz, Einstein, Klein and the author the four conservation laws of matter (of the energy-momentum tensor) are connected with the invariance of the Action with respect to coordinate transformations, expressed through four independent functions, the electromagnetic conservation law is connected with the new scale-invariance, expressed through a fifth arbitrary function. The manner in which the latter resembles the energy-momentum principle seems to me to be the strongest general argument in favour of the present theory—insofar as it is permissible to talk of justification in the context of pure speculation. (p. 32)

Bearing in mind what we have said so far about Noether's two theorems, can Weyl be right that he has connected conservation of electric charge with local gauge symmetry? In his excellent book on the history of unified field theories, Vizgin (1994) insists:

> In view of the fact that in accordance with Noether's first theorem conservation laws must be associated with finite-parameter continuous transformations, however, it must be recognized that, strictly speaking, neither the energy-momentum conservation law follows from the invariance of the action with respect to arbitrary smooth transformations nor the charge conservation law from gauge invariance. The true symmetry of the charge conservation law was found to be gauge symmetry of the first kind. (p. 96)[19]

So what does Weyl actually do? He begins with the action associated with his unified theory of gravitation and electromagnetism, and an arbitrary variation of the dependent variables of the associated Lagrangian, vanishing on the boundary. The form of the action is not given; what Weyl requires is that, discarding boundary terms,

$$\delta S = \int (W^{\mu\nu}\delta g_{\mu\nu} + w^{\mu}\delta A_{\mu})\,\mathrm{d}x, \tag{14}$$

where $\delta g_{\mu\nu}$ is an arbitrary variation in the metric and δA_{μ} is an arbitrary variation in the electromagnetic vector potential. If we were to set $\delta S = 0$ under each of these arbitrary variations, then we would have two applications of Hamilton's principle, with $W^{\mu\nu} = 0$ and $w^{\mu} = 0$ being the resulting Euler–Lagrange equations. Weyl interprets $W^{\mu\nu} = 0$ as the gravitational field equations and $w^{\mu} = 0$ as the

[17] For a discussion of Weyl's background philosophical motivations, and their connection to Husserlian phenomenology, see Ryckman (2001).

[18] Weyl's 1918 theory is of interest for many reasons, including the issue at stake here (the connection between gauge symmetry and conservation of electric charge). However, it is well-known that Einstein was quick to point out difficulties with the theory (see Vizgin, 1994, p. 98–104; see also Brown & Pooley, 1999, Section 5, for a strengthening of Einstein's critique).

[19] Gauge symmetry of the first kind is global gauge symmetry.

electromagnetic field equations, but again their form is yet to be specified.[20] In (14) $W^{\mu\nu}$ and w^{μ} are therefore the Lagrange expressions (using Noether's terminology, see (1) above) associated with the gravitational and electromagnetic equations respectively.

Weyl's purpose here is not, however, to obtain equations of motion via Hamilton's principle, but rather to investigate the consequences of imposing local gauge invariance on the action S. His next step, therefore, is to demand that the arbitrary variations be infinitesimal gauge transformations depending on the arbitrary function $\rho(x)$, and that the action be invariant under such a gauge transformation ($\delta S = 0$). In Weyl's 1918 theory, a gauge transformation consists of an infinitesimal scale transformation

$$\delta g_{\mu\nu} = g_{\mu\nu}\delta\rho \tag{15}$$

combined with an infinitesimal transformation of the electromagnetic potential

$$\delta A_{\mu} = \partial_{\mu}(\delta\rho). \tag{16}$$

Then, with $\delta S = 0$ and substituting the gauge transformation (15) and (16) into (14), we obtain

$$\delta S = \int \{W^{\mu\nu} g_{\mu\nu}\delta\rho + w^{\mu}\partial_{\mu}(\delta\rho)\}\, \mathrm{d}x = 0, \tag{17}$$

from which follows

$$\int \{W^{\mu\nu} g_{\mu\nu}\delta\rho + \partial_{\mu}(w^{\mu}\delta\rho) - (\partial_{\mu}w^{\mu})\delta\rho\}\, \mathrm{d}x \doteq 0, \tag{18}$$

where once again I use the symbol '$\doteq$' to indicate that we have not assumed any Euler–Lagrange equations of motion in deducing this equality. Discarding the boundary term,

$$W^{\mu\nu} g_{\mu\nu}\delta\rho \doteq (\partial_{\mu}w^{\mu})\delta\rho, \tag{19}$$

hence

$$W^{\mu}_{\mu} \doteq \partial_{\mu}w^{\mu}. \tag{20}$$

This expresses a dependence between the Lagrange expressions associated with the gravitational field equations and the electromagnetic field equations.

In order to derive conservation of electric charge, Weyl now demands that the gravitational field equations are satisfied, i.e. $W^{\mu\nu} = 0$, so that (20) becomes

$$\partial_{\mu}w^{\mu} = 0. \tag{21}$$

[20] In fact, Weyl chooses his Lagrangian to be $L = R^{i}_{jkl}R^{jkl}_{i}$ as 'the most natural Ansatz we can make' for L. Earlier in the paper he constructed the geometrical curvature components R^{i}_{jkl} from considerations of parallel transport and length-preserving transport of a vector. This splits into two parts, $R^{i}_{jkl} = P^{i}_{jkl} - \frac{1}{2}\delta^{i}_{j}F_{kl}$, where $F_{kl} = 0$ characterises the absence of an electromagnetic field (transfer of the magnitude of a vector is integrable) and $P^{i}_{jkl} = 0$ characterises the absence of a gravitational field (transfer of the direction of a vector is integrable). As a consequence of choosing this Ansatz, and demanding that $\delta S = 0$ under an infinitesimal gauge transformation, Weyl recovers Maxwell's equations, but not Einstein's. Weyl's gravitational equations are fourth-order (see Weyl, 1918a, in O'Raifeartaigh, 1997, p. 33–34).

He then inserts the Lagrange expression associated with the inhomogeneous Maxwell equations[21]

$$w^{\mu} = \partial_{\nu} F^{\mu\nu} - J^{\mu} \tag{22}$$

giving

$$\partial_{\mu}(\partial_{\nu} F^{\mu\nu} - J^{\mu}) = 0. \tag{23}$$

Then, since the antisymmetry of $F^{\mu\nu}$ guarantees that $\partial_{\mu}\partial_{\nu}F^{\mu\nu} \doteq 0$, we obtain

$$\partial_{\mu} J^{\mu} = 0 \tag{24}$$

as desired. Notice that this derivation does not involve demanding that the Maxwell equations are satisfied. Instead, it relies on the gravitational field equations being satisfied (but *not* on the explicit form of those equations), and on the fact that the gravitational equations and the Maxwell equations are not independent of one another (this lack of independence being a consequence of imposing local gauge invariance). In other words, there is some redundancy in the total set of field equations: the conservation law for electric charge can be obtained either from the Maxwell equations directly, or via the Maxwell–Lagrange expression (22) and the gravitational field equations.

Having followed a similar derivation for the four energy-momentum conservation laws,[22] Weyl (1918a; p. 33 of the translation in O'Raifeartaigh, 1997) writes:

> The five conservation laws can be eliminated from the field equations since they are obtained in two ways and thereby show that five of the field equations are superfluous.

This is Weyl's route to the conservation laws. Clearly, the means by which he connects conservation of electric charge with gauge symmetry is distinct from the routes in the modern literature, discussed in Section 3 above. We will compare these methods in Section 5, but in order to make the comparison precise we first need to look at the relationship between Weyl's work and Noether's theorems.

4.2. *Weyl's 1918 theory and Noether's second theorem*

Weyl's derivation is essentially an application of Noether's second theorem. In Noether's second theorem we throw away the boundary terms, as Weyl does, and we get Eq. (6), where a_{ki} and b^{μ}_{ki} are given by Eq. (7). For Weyl's theory, our symmetry transformation depends on the arbitrary function $\rho(x)$, and in infinitesimal form we have (15) and (16). Consider first $E_1 = W^{\mu\nu}$. The Lagrange

[21] One might wonder where Weyl gets the Maxwell–Lagrange expression (22) from. In the 1918 paper, Weyl simply helps himself to this, but in the third edition of *Space Time Matter* (see Weyl, 1952, pp. 287–289) Weyl arrives at this expression via a method which was later used independently by Utiyama (1956, 1959) to derive a theorem applying to all theories with a local gauge-type symmetry structure (see Brading & Brown, 2001).

[22] This parallel derivation is discussed in Brading and Brown (2001).

expression $W^{\mu\nu}$ depends on the metric and so is affected by the infinitesimal transformation of the metric $\delta g_{\mu\nu} = g_{\mu\nu}\delta\rho$. So, $\delta_0\psi_1 = g_{\mu\nu}\delta\rho$ and we have a contribution to only the left-hand side of Noether's second theorem:

$$E_1 a_1 = W^{\mu\nu} g_{\mu\nu} = W^{\mu}_{\mu} \tag{25}$$

(where we drop the k-index since the transformation depends on only one arbitrary function $\rho(x)$). Now consider $E_2 = w^{\mu}$. The Lagrange expression w^{μ} depends on the vector potential A_{μ} and so is affected by the infinitesimal transformation of the vector potential $\delta A_{\mu} = \partial_{\mu}(\delta\rho)$. So, $\delta_0\psi_2 = \partial_{\mu}(\delta\rho)$, and we have a contribution to only the right-hand side of Noether's second theorem:

$$\partial_{\mu}(E_1 b^{\mu}_1) = \partial_{\mu} w^{\mu}. \tag{26}$$

Therefore, equating (25) and (26), Noether's second theorem gives us

$$W^{\mu}_{\mu} \stackrel{\circ}{=} \partial_{\mu} w^{\mu},$$

exactly as Weyl showed in Eq. (20).

Therefore, Weyl's 1918 connection between local gauge invariance and conservation of electric charge begins from an instance of Noether's second theorem. He then simply assumes that $W^{\mu}_{\mu} = 0$, along with the form of the Maxwell–Lagrange expression, and this allows him to complete his derivation (see Section 4.1 above). This clarifies the relationship between Weyl's 1918 work and Noether's 1918 work.

4.3. *Weyl's 1928–29 work and Noether's second theorem*

In his 1929 paper 'Electron and Gravitation' Weyl follows exactly the same general strategy as in his 1918 work, applying it to his new unified theory of matter and electromagnetism (as opposed to the 1918 unified theory of gravity and electromagnetism). He requires that the variation in the action under a local gauge transformation be zero. Discarding boundary terms this gives us a relation between the Lagrange expressions for the matter fields and the Lagrange expressions for the electromagnetic fields. If we then assume that the electromagnetic equations of motion are satisfied, we are left with a continuity equation from which conservation of charge can be derived.[23] Weyl's approach to charge conservation in his 1928 book is slightly different, but once again there is nothing that relies on global gauge invariance or that resembles an application of Noether's first theorem. He first discusses conservation of electric charge as a consequence of the field equations of matter (p. 214), and only at the end of the section mentions his standard approach. Here, he states (Weyl, 1928),

> The theorem of the conservation of electricity follows, as we have seen, from the equations of matter, but it is at the same time a consequence of the electromagnetic equations. The fact that [conservation of electricity] is a consequence of both sets of field laws means that these sets are not independent,

[23] For details, see pp. 140–141 of the translation of Weyl (1929) in O'Raifeartaigh (1997).

> i.e. that there exists an identity between them. The true ground for this identity is to be found in the gauge invariance... (p. 217)

He then sketches his standard derivation, the derivation that is essentially an application of Noether's second theorem.

In order to see how a conservation law results from the dependencies of Noether's second theorem, it is useful to look at the details of an example. We will do this below in Section 6, where we apply Noether's second theorem in the context of modern relativistic field theory.

5. Weyl and relativistic field theory

We are now in a position to clarify the relationship between Weyl's work and the standard modern connection between gauge invariance and conservation of electric charge, summarising what has been shown in the preceding sections.

Weyl's 1918 connection between gauge invariance and conservation of charge is usually thought to come to fruition in relativistic field theory, and in particular through Weyl's own re-application of his 1918 ideas in his 1928–29 work. It is true that the connection between conservation of electric charge and gauge symmetry was first suggested by Weyl, and that he re-applied it in a new context in 1928–29. It is also true that this connection now has an established place in modern physics. However, it is not true that the connection in modern physics is made in the same way as Weyl made it. The standard account appeals to global gauge invariance and Noether's first theorem, yet Weyl never used Noether's first theorem. Rather, he used local gauge invariance and (what we have now shown to be an instance of) Noether's second theorem.

6. Relativistic field theory and Noether's second theorem

There remains one arm of our three-way relationship which is in need of clarification: the application of Noether's second theorem in relativistic field theory.

Recall the Lagrangian for a complex scalar field interacting with an electromagnetic field, (see Eq. (11)). This Lagrangian has been constructed to be invariant under local gauge transformations (10), as we have discussed in Section 3. If we apply Noether's second theorem (6), we find that ψ and ψ^* give a contribution to the left-hand side of (6), and the A_μ give a contribution to the right-hand side, as we will now see.

Consider first the gauge transformation of ψ and ψ^* (see (10)). Infinitesimally, $\delta_0\psi = -\mathrm{i}q(\delta\theta)\psi$ and $\delta_0\psi^* = iq(\delta\theta)\psi^*$. Therefore, the contribution of these fields to Noether's second theorem is entirely to the left-hand side of (6), which becomes

$$\left[\frac{\partial L}{\partial\psi} - \partial_v\left(\frac{\partial L}{\partial(\partial_v\psi)}\right)\right](-\mathrm{i}q\psi) + \left[\frac{\partial L}{\partial\psi^*} - \partial_v\left(\frac{\partial L}{\partial(\partial_v\psi^*)}\right)\right]\mathrm{i}q\psi^*. \tag{27}$$

The contribution of the A_μ, on the other hand, is entirely to the right-hand side of (6), since $\delta A_\mu = \partial_\mu(\delta\theta)$, and that side reads

$$\partial_\mu\left[\frac{\partial L}{\partial A_\mu}-\partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right)\right]. \tag{28}$$

Noether's second theorem therefore delivers the following Noether identity:

$$\left[\frac{\partial L}{\partial\psi}-\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\right)\right](-\mathrm{i}q\psi)+\left[\frac{\partial L}{\partial\psi^*}-\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi^*)}\right)\right]\mathrm{i}q\psi^* \\ \mathrel{\hat{=}}\partial_\mu\left[\frac{\partial L}{\partial A_\mu}-\partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right)\right]. \tag{29}$$

In other words, the theorem tells us that not all the Lagrange expressions are independent of one another, and gives us the interdependency.

There are various ways to proceed from here. Straightforward substitution of the Lagrangian L_{total} into (29) yields

$$\partial_\mu\partial_\nu F^{\mu\nu} \mathrel{\hat{=}} 0. \tag{30}$$

This can be found in the literature (see for example Kastrup, 1987; Byers, 1999), where the standard claim is that Noether's second theorem leads to 'Bianchi-type identities' such as (30), so-called after their analogue in General Relativity (see Brading & Brown, 2001, where further discussion of the significance of Noether's second theorem and of the 'Bianchi identities' can be found). For our present purposes, however, there is something more interesting that we can do. We can return to (29), and follow Weyl's procedure of demanding that one set of Euler–Lagrange equations is satisfied. Suppose we assume that the equations of motion for A_μ are satisfied. Then, from (29),

$$\left[\frac{\partial L}{\partial\psi}-\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\right)\right](-\mathrm{i}q\psi)+\left[\frac{\partial L}{\partial\psi^*}-\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi^*)}\right)\right]\mathrm{i}q\psi^*=0, \tag{31}$$

and substituting into L_{total} we obtain

$$\partial_\mu j^\mu=0, \tag{32}$$

where j^μ is the conserved current derivable via Noether's first theorem (13).

Thus, as in Weyl's original case, the interdependence between the two sets of field equations reveals itself in a conservation law. In this case, given local gauge invariance of the Lagrangian, satisfaction of the electromagnetic field equations is related to a restriction on the sources, i.e., that electric charge is conserved.[24] Notice

[24] We could also follow Weyl's general procedure by starting from (29) and assuming that the Euler–Lagrange equations for ψ and ψ^* are satisfied. Then, the left-hand side of (29) becomes zero and we obtain

$$\partial_\mu(\partial_\nu F^{\mu\nu}+j^\mu)=0.$$

But since we know that $\partial_\mu j^\mu=0$ when ψ and ψ^* satisfy the Euler–Lagrange equations, we have that

$$\partial_\mu\partial_\nu F^{\mu\nu}=0.$$

However, following Weyl's procedure in this case is misleading because the validity of the conclusion does not depend on the Euler–Lagrange equations for ψ and ψ^* being satisfied. Rather, it is a consequence of the anti-symmetry of the $F^{\mu\nu}$ term, and can be extracted using Noether's second theorem when no Euler–Lagrange equations are assumed to be satisfied (as we saw earlier in this section).

that this restriction does *not* depend on the *form* of the electromagnetic Euler–Lagrange equations, but only on the assumption that these equations are satisfied, whatever their explicit form may be.

7. Proper and improper conservation laws

In Section VI of her paper, Noether refers to a distinction made by Hilbert, a distinction which she claims is clarified by her work. In theories prior to general relativity, such as classical mechanics and electrodynamics, the conservation laws are consequences of the equations of motion of the associated particles or fields. Hilbert contrasted this with general relativity, remarking that here the conservation of energy of the matter fields can be obtained without the matter field equations being satisfied. In Noether's terminology, conservation of energy in general relativity is an 'improper' conservation law, because it follows from satisfaction of the Einstein field equations, independently of the specific form of those equations and of the Euler–Lagrange equations associated with the matter fields. The distinction between proper and improper conservation laws, and the case of general relativity, are discussed in detail in Brading and Brown (2001) and also in Trautman (1962). Here, we simply note that conservation of electric charge in locally gauge invariant relativistic field theory is an improper conservation law, because it follows from local gauge invariance and the satisfaction of the field equations for A_μ, independently of whether the matter field equations (the field equations for ψ and ψ^*) are satisfied. This is in contrast to the theory associated with the free complex scalar field, described by the Lagrangian L_m (see Section 3, above), which is globally gauge invariant but not locally gauge invariant; in this case, conservation of charge holds only when the Euler–Lagrange equations for ψ and ψ^* are satisfied, and it is therefore a proper conservation law.

8. Which symmetry?

We began with the observation that there is an apparent conflict between the standard treatment of the connection between gauge symmetry and conservation of electric charge in relativistic field theory textbooks, and claims made by Weyl, the father of gauge theory. In the process of addressing this problem, we have clarified the three-way relationship between Weyl's work, Noether's theorems, and modern relativistic field theory. We have used Noether's two theorems to show that there are two routes to conservation of electric charge in locally gauge invariant relativistic field theory: one is the standard route using global gauge invariance and Noether's first theorem, while the other uses local gauge invariance and Noether's second theorem.[25] The latter route is essentially the method used by Weyl in both 1918 and 1928–29. Therefore, although Weyl was the first to make the connection between gauge symmetry and conservation of electric charge, his connection is different from

[25] For a third route, see 'Utiyama's theorem' discussed in Brading and Brown (2001).

that found in modern relativistic field theory textbooks. Although the standard textbook route to conservation of electric charge via Noether's first theorem is correct, it is subtly misleading in locally gauge invariant relativistic field theory, since it implies that conservation of electric charge is dependent upon satisfaction of the equations of motion for the matter fields. In fact, conservation of electric charge can be derived without the matter field equations being satisfied, using local gauge invariance and the satisfaction of the electromagnetic field equations instead. In short, conservation of electric charge in locally gauge invariant relativistic field theory is a consequence of the lack of independence of the matter and gauge fields (itself a consequence of local gauge invariance) rather than simply a consequence of the equations of motion of the matter fields. This understanding of the conservation law is immediately apparent from Noether's second theorem and Weyl's derivation, but it is an insight that gets lost from the standard textbook point of view.

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Appendix. Maxwell electromagnetism—an apparent mystery resolved

This discussion of Maxwell electromagnetism (by which we mean electromagnetism without a gauge-dependent matter field) is included partly for the sake of completeness, partly because it is an example of where failure to appreciate the domain of applicability of the first theorem has led to confusion,[27] and partly

[26] Following the recent death of Professor Lochlainn O'Raifeartaigh, I would like to dedicate this paper to his memory.

[27] For example, Lanczos (1970, Chapter XI, Section 20) seeks to apply 'Noether's principle' to Maxwell electromagnetism in order to derive conservation of electric charge. He is apparently attempting to extend the first theorem to the domain of the second theorem, where he claims that 'Noether's principle' is 'equally valid'. His method involves treating the gauge parameter as an additional field variable; whether or not the derivation is successful, it is certainly not using either of Noether's theorems.

because applying the second theorem to this case gives rise to an apparent mystery (a mystery which is nevertheless quickly dispelled).

For the Lagrangian associated with Maxwell's equations, the gauge transformation consists of a transformation of the vector potential A_μ only:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \theta. \tag{A.1}$$

This means that, unlike in the case of relativistic field theory, there is no non-trivial global subgroup to which Noether's first theorem applies. Only Noether's second theorem is of interest with respect to Maxwell electromagnetism.

Recall Noether's second theorem: if the action S is invariant under a continuous group of transformations depending smoothly on the arbitrary functions $p_k(x)$ and their derivatives, then Eq. (6) holds where $\delta_0\psi_i = a_{ki}\delta p_k + b^\nu_{ki}\partial_\nu(\delta p_k)$. For a Maxwell gauge transformation, with $\psi = A_\mu$, we have $p(x) = \theta(x)$, and so in this case $a_{A_\mu} = 0$ and $b^\nu_{A_\mu} = \delta^\nu_\mu$. The Lagrange expression associated with Maxwell's equations (with sources) is $E_{A_\mu} = \partial_\nu F^{\mu\nu} - J^\mu$. Thus, Noether's second theorem gives us

$$\partial_\gamma\{[\partial_\nu F^{\mu\nu} - J^\mu]\delta^\gamma_\mu\} \triangleq \partial_\mu[\partial_\nu F^{\mu\nu} - J^\mu] \triangleq 0, \tag{A.2}$$

from which we conclude via the anti-symmetry of $F^{\mu\nu}$ that

$$\partial_\mu J^\mu \triangleq 0. \tag{A.3}$$

So it appears at first sight that conservation of electric charge follows from Noether's second theorem (subject to the usual constraints on boundary conditions) for the Maxwell–Lagrange expression, with no requirement that the Maxwell equations be satisfied. On the face of it, the derivation looks rather mysterious: we appear to have derived conservation of electric charge without requiring that *any* equations of motion be satisfied; surely this cannot be right.

The only requirement we have put in is that the Lagrangian be invariant under gauge transformations. The Lagrangian associated with Maxwell electromagnetism (with sources) is:

$$L = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu, \tag{A.4}$$

where J^μ is the four-current, assumed to be a function of position.[28] Applying a gauge transformation, we obtain

$$L' = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu(A_\mu + \partial_\mu\theta) = L + J^\mu\partial_\mu\theta. \tag{A.5}$$

In fact, then, the Lagrangian is not invariant under gauge transformations. However, the extra term picked up makes no difference to the Euler–Lagrange equations for A_μ because the extra term has no dependence on A_μ. We can therefore regard the transformation as a symmetry transformation. However, this does not mean that we can apply Noether's second theorem. Noether's second theorem as stated above requires that the Lagrangian be invariant. In fact, her derivation goes through so long as the Lagrangian is invariant up to a

[28] Substitution of this into the Euler–Lagrange equations yields the Lagrange expression used above: $w^\mu = \partial_\nu F^{\mu\nu} - J^\mu$.

divergence term.[29] Therefore, we can apply Noether's second theorem only if we convert the extra term in the transformed Lagrangian to a divergence term. In other words, we must have that

$$L' = L + \partial_\mu (J^\mu \theta). \tag{A.6}$$

This will only be true if

$$\partial_\mu J^\mu = 0. \tag{A.7}$$

Therefore, the requirement that Noether's second theorem be applicable implicitly embodies the restriction that J^μ be a conserved current.

In short, although the derivation of the conserved current via Noether's second theorem does not involve claiming that the Maxwell field equations are satisfied, it *does* involve the prior assumption that J^μ is conserved. Thus, the apparent mystery dissolves: after all is said and done, we are only getting out what we have already put in.

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[29] See Doughty, 1990; Brading and Brown, 2001.

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