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A Note on General Relativity, Energy Conservation, and Noether's Theorems

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The variational problem posed by Emmy Noether in her seminal 1918 paper leads to three theorems, two of which she presents in that paper and the third of which is due to F. Klein, also in 1918.[1] The origins of these theorems lie in the discussions of Klein, Noether, D. Hilbert and A. Einstein over the status of energy conservation in generally covariant theories such as General Relativity. In this paper I will outline one thread of this discussion and show how the three theorems of Noether and Klein can be brought to bear. The particular thread of interest begins with Klein's observation (in his response to Hilbert's (1916) first note on the foundations of physics) that the energy conservation law associated with Hilbert's energy vector is a mathematical identity, in constrast to the familiar energy conservation laws of mechanics which are not identities.[2] These two aspects—the claim that energy conservation is an identity, and the claim that this marks a contrast with other theories—are picked up by Hilbert and by Einstein, and are the subject of this note.

8.1 Historical Background

Klein's 1917 response to Hilbert[3] includes a section specifically on Einstein's theory (Klein 1917, 476–477, comment 9) in which he considers the energy conservation law found in Einstein's 1916 paper "Die Grundlagen der allgemeinen Relativitätstheorie," consisting of the vanishing of the divergence of two terms:

$$\partial_\nu (T_\sigma^\nu + t_\sigma^\nu) = 0 \tag{8.1}$$

where T_σ^ν and t_σ^ν are the so-called energy components associated with the electromagnetic and gravitational fields, respectively. Using the field equations, the "energy," can be re-written as:

$$T_\sigma^\nu + t_\sigma^\nu = -\partial_\rho \left(\frac{\partial G^*}{\partial g^{\mu\sigma}_{,\rho}} g^{\mu\nu} \right) \tag{8.2}$$

where G^* is the gravitational Lagrangian depending on the $g^{\mu\nu}$ up to first derivatives only (see Einstein 1916). Einstein and Klein agree that the divergence of the right-hand side of (8.2) vanishes as an identity:

$$\partial_\rho \partial_\nu \left(\frac{\partial G^*}{\partial g^{\mu\sigma}_{,\rho}} g^{\mu\nu} \right) \equiv 0. \tag{8.3}$$

However, for Klein this further implies that (8.1) holds as an identity. He points out the relationship between Einstein's terms T_σ^ν and T_σ^ν and those appearing in his own treatment, and concludes that Einstein's energy conservation law is therefore an "identity."

In 1917 Klein had begun a correspondence with Einstein,[4] and on 13 March 1918 Einstein writes to Klein beginning his letter as follows:[5]

> Highly esteemed Colleague,
> It was with great pleasure that I read your extremely clear and elegant explanations on Hilbert's first note. However, I do not find your remark about my formulation of the conservation laws appropriate. For equation (8.1) is by no means an identity, no more so than (8.2); only (8.3) is an identity.

Klein replies to Einstein immediately (20 March 1918),[6] attempting to clarify his point, the essence of which is that Einstein's conservation law (8.1) can be re-expressed as the divergence of two terms: a term which itself vanishes via the field equations (hence the vanishing of the divergence of this term is "physically meaningless"), and a term whose divergence vanishes identically. Hence the taking of the divergence does not have physical significance. Einstein replies on 24 March 1918,[7] writing that he "does not concede" that either Klein's relations or his (the relations (8.1)) are "devoid of content." Rather, he says, "What they contain is *a part* of the content of the field equations."[8]

After this letter from Einstein the correspondence on this issue slows down, but Klein continues to work on it with the assistance of Noether. In 1918 Noether and Klein each publish papers that together contain three theorems, the result of work that they had been doing together.[9] On 15 July, Klein writes to Einstein with the reasoning found in his 1918 paper that in essence leads to the Boundary theorem (see below). Further details and discussion of the Klein–Einstein correspondence during this period leading to the Noether and Klein papers, and of the crucial role played by Noether, can be found in (Rowe 1999, see especially pp. 212–28). The content of these papers enables us to resolve both aspects of the story mentioned above, but first let us mention the historical background to the second aspect.

In his reply to Klein, Hilbert (Klein 1917, 477–482) agrees with Klein,[10] and goes further, postulating that conservation of energy holding "identically" is *characteristic* of any generally covariant theory. He writes:[11]

"With your considerations on the energy theorem I am in full factual agreement: with Emmy Noether, whose help I called upon for clarification of questions pertaining to the analytical treatment of my energy theorem more than a year ago, I found accordingly that the energy components set up by me, just as those of Einstein, can be formally transformed by means of the Lagrangian differential equations ... of my first contribution, into expressions whose divergence *identically*, that is without reference to the Lagrangian equations [...] vanishes.

"Since on the other hand the energy equations of classical mechanics, of the theory of elasticity, and of electrodynamics, are fulfilled only as a consequence of the Lagrangian differential equations of these problems, then it is justified if you accordingly do not recognise in my energy equations the analogues of those of your theory. Certainly I maintain that for *general* relativity, that is, in the case of general invariance of the *Hamiltonian* function, [such] energy equations ... in general do not exist ... I might designate this circumstance as a characteristic trait of the general theory of relativity. For my assertion, mathematical proof should be adduced."

Once again, Einstein is in disagreement with Hilbert and Klein. In his letter to Klein of 13 March 1918, Einstein insists that

"The relations here are exactly analogous to those for nonrelativistic theories."

As we shall see below, Noether's 1918 paper is explicitly concerned with giving the mathematical proof that Hilbert sought for his claim.

8.2 Discussion[12]

We now turn our attention to how to resolve these two related disagreements between Einstein, Hilbert and Klein, using results based on the 1918 papers of Klein and Noether entitled "On the differential laws for conservation of momentum and energy in Einstein's theory of gravitation" and "Invariant variation problems," respectively.

Klein's paper is concerned with results that follow for generally covariant theories, and in particular General Relativity.[13] The diffeomorphism freedom of General Relativity is a local symmetry in the sense that the symmetry depends on arbitrary functions of space and time. In its generalised form (i.e., applying to all Lagrangian theories that have a local symmetry), we call the theorem contained in Klein's paper the "Boundary theorem" for reasons to do with how it is derived (see Brading and Brown, 2003b). We can state this theorem as follows.

8.2.1 Boundary Theorem

If a continuous group of transformations depending smoothly on ρ arbitrary functions of time and space $p_k(x)(k = 1, 2, \ldots, \rho)$ and their first derivatives is a Noether symmetry[14] group of the Euler–Lagrange equations associated with $L(\varphi_i, \partial_\mu\varphi_i, x^\mu)$, then the following three sets of ρ relations are satisfied, one for every parameter on which the symmetry group depends:

$$\sum_i \partial_\mu \left\{ \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki} \right\} = \partial_\mu j^\mu_{k(\text{Noether})} \tag{8.4}$$

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki} = j^\mu_{k(\text{Noether})} - \sum_i \left\{ \partial_\nu \left(\frac{\partial L}{\partial(\partial_\nu \varphi_i)} b^\mu_{ki} - \frac{\partial(\Delta\Lambda^\mu)}{\partial(\partial_\nu \Delta p_k)} \right) \right\}, \tag{8.5}$$

$$\sum_i \left\{ \left(\frac{\partial L}{\partial((\partial_\mu \varphi_i)} b^\nu_{ki} - \frac{\partial(\Delta\Lambda^\nu)}{\partial(\partial_\mu \Delta p_k)} \right) - \left(\frac{\partial L}{\partial(\partial_\nu \varphi_i)} b^\mu_{ki} - \frac{\partial(\Delta\Lambda^\mu)}{\partial(\partial_\nu \Delta p_k)} \right) \right\} = 0 \tag{8.6}$$

where the infinitesimal transformation $\delta_0 \varphi_i$ is given by

$$\delta_0 \varphi_i = \sum_k \left\{ a_{ki}(\varphi_i, \partial_\mu \varphi_i, x) \Delta p_k(x) + b^\nu_{ki}(\varphi_i, \partial_u \varphi_i, x) \partial_\nu \Delta p_k(x) \right\}, \tag{8.7}$$

Δp_k indicating that we are considering infinitesimal transformations, the a_{ki} and b^μ_{ki} depending on the particular transformation in question, and $j^\mu_{k(\text{Noether})}$ is the "Noether current"[15] associated with the kth arbitrary function:

$$j^\mu_{k(\text{Noether})} := -\sum_i \left\{ \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \frac{\partial(\delta_0 \varphi_i)}{\partial(\Delta p_k)} + L \frac{\partial(\delta x^\mu)}{\partial(\Delta p_k)} - \frac{\partial(\Delta\Lambda^\mu)}{\partial(\Delta p_k)} \right\}. \tag{8.8}$$

The terms in Λ^μ occur when the action associated with the Lagrangian L is not strictly invariant under the transformations being considered, instead picking up a divergence term. This is the case for the so-called Einstein $\Gamma\Gamma$ action, for example.[16] The above three identities (8.4)–(8.6), along with that of Noether's second theorem (see below), are not independent of one another, but we present all four here since that is how they emerged historically.

Rearranging the first identity of the Boundary theorem, equation (8.4), we get:

$$\partial_\mu \left\{ j^\mu_{k(\text{Noether})} - \left(\frac{\partial L}{\partial \varphi_i} - \partial \nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki} \right\} = 0. \tag{8.9}$$

Hence, defining

$$\Theta^\mu_k := j^\mu_{k(\text{Noether})} - \left(\frac{\partial L}{\partial \varphi_i} - \partial \nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki}, \tag{8.10}$$

we have that

$$\partial_\mu \Theta^\mu_k = 0 \tag{8.11}$$

holds identically. From this, we infer the existence of the so-called "superpotentials" $U^{\mu\nu}_k$, such that

$$\Theta^\mu_k = \partial_\nu U^{\mu\nu}_k, \tag{8.12}$$

where

$$\partial_\mu \partial_\nu U^{\mu\nu}_k = 0 \tag{8.13}$$

holds identically. These mathematical manipulations allow us to re-write the Noether current in the following form:

$$j^{\mu}_{k(\text{Noether})} = \left(\frac{\partial L}{\partial \varphi_i} - \partial \nu \frac{\partial L}{\partial (\partial_\nu \varphi_i)} \right) b^{\mu}_{ki} + \partial_\nu U^{\mu\nu}_k. \tag{8.14}$$

In other words, the Noether current can be expressed as consisting of a term which vanishes when the field equations are satisfied,

$$\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \varphi_i)} = 0, \tag{8.15}$$

and a term whose divergence vanishes identically.

Now consider the conservation law[17]

$$\partial_\mu j^{\mu}_{k(\text{Noether})} = 0. \tag{8.16}$$

Given that the Noether current can be re-written in the form (8.14), we see that (8.16) can be understood as the vanishing of the divergence of two contributions. The first contribution vanishes when the field equations are satisfied without any need to take the divergence; the divergence of the second contribution vanishes identically. We can therefore re-express Klein's concern over the status of Einstein's conservation law as a more general point about conservation laws for Noether currents associated with local symmetries (i.e., where the k subscript relates to an arbitrary function of space and time p_k). The Kleinian claim is that because we can re-write the Noether current in the above form, the taking of the divergence does not lead to a physically significant result; the conservation law (8.16) therefore lacks physical significance.

At least a part of Einstein's response seems to be that (8.16) holds only when the field equations are satisfied, and that we are therefore making use of physically significant information in order to move from (8.14) to (8.16). This is true, but it doesn't address the full weight of the problem: the term of the Noether current involving the Euler–Lagrange equations vanishes on-shell *without* any need to take the divergence of the Noether current. Taking the divergence plays a role only with respect to the second term, and there the divergence vanishes identically. We are back to the question: wherein lies the physical content in taking the divergence of the Noether current and finding that the resulting expression vanishes?

I think that the right thing to say at this point is as follows. We have shown that whenever we have a local symmetry, the associated Noether current can be re-written in the form (8.14) such that when the field equations are satisfied

$$j^{\mu}_{k(\text{Noether})} = \partial_\nu U^{\mu\nu}_k. \tag{8.17}$$

Part of the Kleinian worry is that the associated continuity equation for $j^{\mu}_{k(\text{Noether})}$ lacks physical content because of (8.13). But notice: while it is true that we can always write an expression of the form (8.17) when the field equations are satisfied, there remains the question of whether, and if so when, this equation expresses a physically significant relation. So far in doing the re-writing all we have done is mathematics, and only mathematics. The relation (8.17) gains *physical* significance only when it holds "not as an identity or definition, but as a field equation postulated to relate two separate systems" (Deser 1972, p. 1082). Consider, for example, the Maxwell field equations

$$J^{\mu} = \partial_{\nu} F^{\mu\nu}. \tag{8.18}$$

These equations are of the form (8.17), and

$$\partial_{\mu}\partial_{\nu} F^{\mu\nu} = 0 \tag{8.19}$$

holds simply in virtue of the antisymmetry of $F^{\mu\nu}$. Nevertheless, we do not say that conservation of electric charge is a mathematical identity without physical significance. This is because the equations (8.18) are not a mere mathematical re-expression of the current J^{μ}; they express a physically significant relation between two different types of field: on the left-hand side we have a current, J^{μ}, depending on the matter fields carrying the electric charge, and on the right-hand side we have an expression depending on the electromagnetic fields, $F^{\mu\nu}$. Thus, the current conservation law follows via (8.18) and (8.19), and since (8.18) is physically significant so is the current conservation law.

Similarly in the case of General Relativity, the re-expression of energy-momentum through a relation of the form (8.17) has physical content because it gives a relation between the behaviour of the metric and the matter fields, it is a field equation with physical content, and hence the conservation law that follows from it (via an identity for the right-hand side) also has physical content.

This is, I believe, how we should understand the first aspect of the story, concerning the claim that energy conservation is an identity. Turning now to the second aspect, the contrast with other theories alleged by Hilbert and disputed by Einstein, we need to look at Noether's paper (Noether 1918). In that paper Noether proved two theorems, the first holding with respect to the global symmetries of a theory, and the second holding with respect to local symmetries. We may state these two theorems as follows.[18]

Noether's First Theorem

If a continuous group of transformations depending smoothly on ρ constant parameters ω_k $(k = 1, 2, \ldots, \rho)$ is a Noether symmetry group of the Euler–Lagrange equations associated with $L(\varphi_i, \partial_{\mu}\varphi_i, x^{\mu})$, then the following ρ relations are satisfied, one for every parameter on which the symmetry group depends:

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_{\mu} \frac{\partial L}{\partial(\partial_{\mu}\varphi_i)} \right) \frac{\partial(\delta_0 \varphi_i)}{\partial(\Delta\omega_k)} = \partial_{\mu} j^{\mu}_{k(\text{Noether})}, \tag{8.20}$$

where $\Delta\omega_k$ indicates that we are taking infinitesimal symmetry transformations,

$$\delta_0 \varphi_i = \frac{\partial(\delta_0 \varphi_i)}{\partial(\Delta\omega_k)} \Delta\omega_k, \tag{8.21}$$

and where $j^{\mu}_{k(\text{Noether})}$ is the Noether current (8.8), the arbitrary functions p_k replaced by the arbitrary parameters ω_k.

Noether's first theorem is widely known for the general connection it makes between symmetries and conservation laws. When the left-hand side of (8.20) vanishes (for example via the field equations, but see Brown and Brading (2002), for a more detailed discussion) we arrive at a conservation law (8.16). This was not the main purpose of her paper, however. Rather, Noether was providing the proof that Hilbert has asked for concerning his conjecture, and for that we need also her second theorem.

Noether's Second Theorem

If a continuous group of transformations depending smoothly on ρ arbitrary functions of time and space $p_k(x)$ $(k = 1, 2, \ldots, \rho)$ and their first derivatives is a Noether symmetry group of the Euler–Lagrange equations associated with $L(\varphi_i, \partial_\mu\varphi_i, x^\mu)$, then the following ρ relations are satisfied, one for every parameter on which the symmetry group depends:

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) a_{ki} = \sum_i \partial_\nu \left\{ b^{\nu}_{ki} \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) \right\} \tag{8.22}$$

where the infinitesimal transformation $\delta_0\varphi_i$ is given by (8.7), above.

As we saw in Section 8.1 above, Hilbert's conjecture was that the difference between generally covariant theories such as General Relativity, and earlier theories such as classical mechanics, can be characterised by the differing status of energy conservation: in generally covariant theories the energy conservation law can be re-written, using the Euler–Lagrange equations, such that it holds "identically." The final section of Noether's paper concerns this "Hilbertian assertion" quoted above (see section 1). She writes:[19]

> From the foregoing we finally obtain the proof of a Hilbertian assertion concerning the connection between the lack of proper energy theorems and "general relativity," and this even in a generalized group-theoretic version.

Where Hilbert uses the term "identically," we shall mean that the current conservation law can be re-written in the form (8.14), this being what we concluded above based on the clarifications made by Klein. The proof then proceeds as follows. In theories that *do not* admit a local symmetry group, only Noether's first theorem (and not her second) can be obtained. In such theories, we apply Noether's first theorem to a global symmetry and obtain a corresponding relation of the form (8.20) from which we may proceed to a current conservation law.[20] However, in theories that admit a local symmetry group we can do two things: the first theorem can be applied to the global subgroup, from which we may proceed to conservation laws, and since the second theorem also applies we can combine it with the first theorem to arrive at what Earman has called "Noether's third theorem."[21] We equate the left-hand sides of the equations of the first and second theorem—and the consequence is just the first identity of the Boundary theorem (8.4). In other words, *only* when the global symmetry group is a subgroup of a local symmetry group can we re-write the Noether current in the form discussed by Klein, i.e., in the form (8.14). In classical mechanics (for example), the global space and time symmetry group is *not* a subgroup of a local symmetry group, so the energy conservation law (associated with global time translations) cannot be re-written in the form (8.14). The form (8.14) is indeed characteristic of generally covariant theories, or indeed of any theory with a local symmetry structure. In this way, Noether proved Hilbert's conjecture, and generalised it beyond the case of general covariance and energy conservation to all continuous global and local symmetry groups.[22]

8.3 Conclusions

The subject of this note has been a small historical thread in the long and complex story of the status of energy conservation in General Relativity, concerning two related claims made by Klein and Hilbert: that the energy conservation law is an identity in generally covariant theories, and that this marks a contrast with other (earlier) theories. Both these claims were disputed by Einstein. We have seen how three theorems proved by Noether and Klein can be brought to bear on this disagreement, showing that:

(1) Klein's worry over the physical significance of the energy conservation law in General Relativity was perhaps not adequately addressed by Einstein, even though in the end we side with Einstein against Klein, and

(2) the possibility of re-writing the energy conservation law in the form that so worried Klein does indeed depend upon the *local* symmetry structure of General Relativity.

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Notes

[1]For the variational problem and derivations of the theorems, see Brading and Brown (2003a) and (2003b).

[2]Klein (1918) p. 475.

[3]On Hilbert's first note on the foundations of physics, see Sauer (1999).

[4]See Rowe (1999), pp. 210–213.

[5]Einstein (1998), document 480, pp. 494–5 of the English translation. Equation numbers are ours; in Einstein's letter the numbers are (22), (23) and (24) and refer to the equations appearing in Klein's note.

[6]Einstein (1998), document 487, pp. 503–507 of the English translation.

[7]Einstein (1998), document 492, pp. 512–514 of the English translation.

[8]Einstein then goes on to give reasons in favour of his own version of the divergence relations rather than Klein's, but the difference between the two does not concern us here.

[9]Noether's paper was originally submitted to the Göttingen Society by Klein in January 1918. She continued to work on it, presenting it to the Society in July and finishing the paper by the end of September (see Rowe, 1999, p. 221).

[10]The friendly tone of this exchange masks the deep criticisms that Klein was making of Hilbert's work (see Rowe, 1999, p. 212).

[11]Hilbert's answer to Klein (1917), p. 477. Thanks to Tilman Sauer and to Tom Ryckman for translating this passage.

[12]The following discussion is reproduced in its essentials in Brading and Brown (2003a).

[13]Section 7 of Klein (1918) is about the relationship between Einstein's formulation of the conservation theorems and Klein's derivations.

[14]A "Noether symmetry" is a symmetry of the field equations that satisfies the requirement that the change in the action arising from an infinitesimal symmetry transformation is at most a surface term. See Brading and Brown (2003b).

[15]See Noether's first theorem, below.

[16]For further details and explanation, and for the derivation of the Boundary theorem, see Brading and Brown (2003b), where references to related results can also be found.

[17]More precisely, this is a continuity equation, and in physics (as opposed to mathematics) the term 'conservation law' is often reserved for expressions of the form $\frac{d}{dt}Q = 0$, where Q is here a conserved charge, these being obtained from continuity equations subject to certain conditions (see Brading and Brown, 2003b).

[18]For the derivations see Barbashov and Nesterenko (1983); Brading and Brown (2003b), and Trautman (1962).

[19]Noether 1918, p. 253–4, p. 201 of the English translation (Tavel) but amended translation (my thanks to Bjoern Sundt and Tom Ryckman).

[20]Note that there is no guarantee that the result is interesting—see Brading and Brown (2003b).

[21]See Earman (2003).

[22]Picking up on Hilbert's use of the term "proper" for the energy conservation laws in non-generally covariant theories, Noether terms such relations "improper." The origins and significance of this terminology in Hilbert's work is the subject of ongoing joint work with Tom Ryckman.