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Symmetries and Noether's theorems

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1 Introduction

Emmy Noether's greatest contributions to science were in algebra, but for physicists her name will always be remembered for her paper of 1918 on an invariance problem in the calculus of variations.[1] The most celebrated part of this work, associated with her 'first theorem', has to do with the connection between continuous (global) symmetries in Lagrangian dynamics and conservation principles, though the main focus of the paper was the relationship between this and the second part of her paper, where she gives a systematic treatment of the more subtle and general case of continuous *local* symmetries (symmetries depending on arbitrary functions of the spacetime coordinates).

The connection between global or 'rigid' symmetries and conservation principles in classical mechanics was hardly news in 1918. As Kastrup (1987) discusses in his historical review, it had been appreciated in the previous century by Lagrange, Hamilton, Jacobi, and Poincaré, and an anticipation of Noether's first theorem in the special cases of the 10-parameter Lorentz and Galilean groups had been given by Herglotz in 1911 and Engel in 1916, respectively. Noether's own contribution is often praised for its degree of generality, and not without reason. But interestingly it does not cover the cases in which the symmetry transformation preserves the Lagrangian or Lagrangian density only up to a divergence term. It does not therefore cover such cases as the boost symmetry in classical pre-relativistic dynamics, although modern treatments of Noether's first theorem commonly rectify this defect.

In this connection it is worth noting that the problem originally addressed by Noether involves specifying the conditions under which a specific class of infinitesimal transformations of the independent and/or dependent variables leaves

[1] Noether's paper was presented to the Royal Society of the Sciences in Göttingen on 7 July 1918 by Felix Klein. See Kastrup (1987), who also discusses the historical background to the Noether paper.

the action invariant. There was no explicit attempt on Noether's part to relate this issue to that of dynamical symmetries. Indeed, the word 'symmetry' never appears in the 1918 paper. Though Noether was well aware that the Euler–Lagrange equations of motion – obtained by application of Hamilton's stationarity principle – are unaffected by the addition of a divergence term to the Lagrangian, this simple fact complicates somewhat the connection between the existence of *symmetry transformations* and the *variational* properties of the action. Even today, texts on general relativity, for instance, sometimes give the impression that the gravitational action, whether or not it contains variables related to matter fields, *must* be invariant under arbitrary coordinate transformations, i.e. that the Lagrangian density must be a scalar density – despite the counterexample offered by Einstein as early as 1916![2] However, it has long been a commonplace in the specialist literature on Noether-type theorems that symmetries are not always connected with transformations which leave the action strictly invariant. What is still perhaps open to discussion is how this fact should be understood. In this respect a recent detailed account of Noether's first theorem in the excellent textbook on Lagrangian dynamics by Doughty (1990), which follows Hill's classic 1951 analysis, is quite at odds with the 1962 treatment by Trautman, for example. In our discussion of the connection between the existence of symmetries and Noether's variational problem (see sections 2 and 3), we will come down on the side of Trautman and others who have defended the form-invariance rather than the 'scalarity' of the Lagrangian in the case of symmetries.

In sections 4 and 5 we state and comment on the three theorems that follow from the Noether variational problem. The main point that we wish to emphasize is this: the three theorems are mathematical tools that enable us to explore and extract the structural properties of our theories that are associated with symmetries. The results may be interesting, even surprising perhaps, but there is no quick route to physics here: any physical significance attaching to the results obtained is tied entirely to the physical significance of the assumptions that are put in, and this significance is something to which the theorems themselves are entirely blind. In the section on Noether's first theorem we discuss how it is that gauge symmetry, sometimes thought of as a freedom in description that is 'merely mathematical',[3] can be connected to a physically significant result such as conservation of electric charge. Noether's second theorem shows how underdetermination will arise for any theory with a local symmetry, while the third theorem (the Boundary theorem) shows in detail the tight restrictions on the possible form of any theory with a local symmetry. These theorems are mathematical tools that may be brought to bear on

[2] For such general relativity texts, see Stephani (1990, p. 95), and Misner, Thorne, and Wheeler (1973, p. 503). The 1916 Einstein counterexample is discussed below.

[3] See Martin, section 4.1, this volume.

interpretational problems; our aim here is to present the theorems in an accessible form, and to offer some brief interpretational remarks along the way.

The final aspect of this paper concerns the historical origins of these theorems. The stimulus to Noether's paper, and to a related paper by Klein, had to do with concerns that the Göttingen mathematicians Hilbert and Klein had surrounding the significance of energy conservation in generally covariant gravitational theories based on Einstein's idea of a dynamical metric field in spacetime. Their claim is that such conservation laws lack physical content (in contrast to the situation in classical mechanics, for example), something Einstein contested. In section 5 of this paper we show how the work done by Noether and Klein bears on the formulation of this issue, and how it should be resolved.

2 Noether's variational problem

From the point of view of physics, Noether's paper concerns theories that can be given a Lagrangian formulation. This covers a wide range of theories, from particle mechanics to general relativity. In the case of relativistic field theories it is standard to make use of Lagrangian densities; these depend on the independent variables x^μ ($\mu = 0, 1, 2, 3$) and the dependent variables (the fields) $\varphi_i(x)$ ($i = 1, \ldots, N$) and their derivatives:[4]

$$L = L(\varphi_i, \partial_\mu \varphi_i, x^\mu). \tag{1}$$

The action, S, is given in terms of the Lagrangian density by:

$$S = \int_R L\, d^4x, \tag{2}$$

where the compact spacetime region R is bounded by initial and final space-like surfaces. The equations of motion for a given field φ_k are the Euler–Lagrange equations; these are derived from the action using *Hamilton's Principle*, according to which the first-order functional variation in the action, resulting from arbitrary infinitesimal transformations of φ_k, vanishes for arbitrary regions of integration, where the variations in φ_k are stipulated to vanish on the boundary of the region of integration. The necessary condition for satisfaction of this variational principle is that φ_k satisfies Euler–Lagrange equations:

$$\frac{\partial L}{\partial \varphi_k} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_k)} = 0. \tag{3}$$

In this and what follows we use the Einstein convention to sum over Greek indices, all other summations being shown explicitly. We will refer to the left-hand side of (3)

[4] The restriction to the first derivative of the fields is for convenience; nothing of principle hangs on it and the generalization to higher derivatives is straightforward.

as the 'Euler expression'. (Notice that not all the fields on which a given Lagrangian density depends may satisfy this principle – such fields are not 'dynamical' in the sense that they do not satisfy Euler–Lagrange equations.)

The variational problem addressed by Noether in her 1918 paper, on the other hand, may be posed as follows. If the first-order functional variation of the action – involving a specific smooth infinitesimal transformation of the independent and/or dependent variables (a transformation which may not vanish on the boundary) – vanishes for an arbitrary region of integration, what general conditions must the variables satisfy?

The first-order functional variation of the action takes the following form:[5]

$$\delta S = \int_R \sum_i \left\{ \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) \delta_0 \varphi_i + \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi_i)} \delta_0 \varphi_i + L \delta x^\mu \right) \right\} d^4x, \tag{4}$$

where the first term on the right-hand side is an 'interior' term arising from the 'bulk' contributions to the variation, and the second term may be thought of as a 'boundary' or 'surface' term since it may be converted into a surface term using Gauss's theorem.[6] The variation $\delta_0\varphi_i$ is the Lie drag $\delta_0\varphi_i = \varphi_i'(x) - \varphi_i(x)$.

Hence, setting $\delta S = 0$ (and recalling that the region of integration is arbitrary), the solution to Noether's 1918 variational problem is:

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) \delta_0 \varphi_i = - \sum_i \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi_i)} \delta_0 \varphi_i + L \delta x^\mu \right). \tag{5}$$

This expresses a restriction on the form of the Lagrangian density that must be satisfied *if* the first-order variation of the action vanishes. The above expression (5) is the crucial result from which Noether's 1918 theorems follow. We will see in section 3 below that when considering the theorems in the context of physics, however, this result is subject to an important generalization in which the strict invariance of the action is relaxed.[7] The theorems that we state in sections 4 and 5 take account of this.

Thus, the variational problem addressed by Noether differs from Hamilton's Principle in four crucial respects. First, it is a *problem* not a *principle*: it asks what follows *if* a given condition is satisfied, rather than requiring that a certain condition be satisfied. Second, the variation considered is a *specific* infinitesimal transformation of the variables, not an arbitrary transformation; and third, the variation considered may involve more than one of the variables, including the independent

[5] For derivations see, for example, Barbashov and Nesterenko (1983), Doughty (1990), and Trautman (1962).

[6] See Schutz (1990, p. 111).

[7] Remarks made by Noether (1918; p. 194 of the English translation) and Klein (1918, section 3 and p. 181, comments (d) and (e)) show that they were aware of the possibility of relaxing the strict invariance of the action integral, but had not yet fully understood the significance of this move.

variables. Finally, the variations are not required to vanish on the boundary of the region of integration.

3 Transformations, invariance of the action, and symmetries in physics

Noether's variational problem is a conditional, where the antecedent is the assumption that the action is invariant under some variation of the dependent and/or independent variables. In this and the following sections we discuss the significance of this assumption and how one should understand it in the context of symmetry transformations in physics. Our target is to link invariance of the action under a variation of the dependent and/or independent variables with infinitesimal symmetry transformations in physics. We begin with general point transformations, and then move to the case of symmetry transformations. The outcome is a revised version of the variational problem addressed by Noether in 1918, appropriate to symmetries in physics, along with the general solution to this problem.

Let us consider a *change* in the action defined by

$$\begin{aligned} \Delta S &= S'[\varphi_i', \partial_\mu' \varphi_i', x'^\mu] - S[\varphi_i, \partial_\mu \varphi_i, x^\mu] \\ &= \int_{R'} L'(\varphi_i', \partial_\mu' \varphi_i', x'^\mu)\, d^4x' - \int_R L(\varphi_i, \partial_\mu \varphi_i, x^\mu)\, d^4x. \end{aligned} \tag{6}$$

Following such a transformation we require that our new Lagrangian density, expressed as a function of the new variables, satisfies Euler–Lagrange equations. In other words, we require that if $S[\varphi_i, \partial_\mu \varphi_i, x^\mu]$ satisfies Hamilton's Principle, then $S'[\varphi_i', \partial_\mu' \varphi_i', x'^\mu]$ also satisfies Hamilton's principle. A sufficient condition for this requirement to be met is that the transformed and untransformed actions differ by at most the integral of a total divergence,

$$\Delta S = \int_R \partial_\mu \Lambda^\mu \, d^4x, \tag{7}$$

since the Euler expression associated with a total divergence vanishes identically.[8] If the transformation is continuous and infinitesimal, then (7) holds when the function Λ is replaced with the infinitesimal function $\Delta\Lambda$.[9] Hence,

$$\Delta S = \int_R \partial_\mu (\Delta \Lambda^\mu)\, d^4x. \tag{8}$$

We now turn our attention to those special transformations that are symmetry transformations.

[8] This is a point noted in passing by Noether (1918; p. 194 of the English translation), but not included in her derivation and statement of her theorems. Allowing for a divergence term gives the most familiar generalization of the results proved by Noether herself, but notice that even this generalization expresses a sufficient *but not necessary* condition on the change in the action; we shall have more to say on this below.

[9] See Doughty (1990, p. 196).

For the purposes of classical theories including field theories prior to second quantization (such as non-relativistic Schrödinger quantum mechanics and relativistic field theories associated with the Klein–Gordon and Dirac equations), the appropriate concept of symmetry in physics is a transformation of the independent and/or dependent variables that leaves the *explicit* form of the equations of motion unchanged. We are concerned with Lagrangian theories, and with the subset of point transformations that leave the explicit form of Euler–Lagrange equations unaffected. A sufficient condition for this is that the transformed Lagrangian density has *the same functional form* as the untransformed Lagrangian density:

$$L'(\varphi_i, \partial_\mu \varphi_i, x^\mu) = L(\varphi_i, \partial_\mu \varphi_i, x^\mu). \tag{9}$$

More generally, the Euler–Lagrange equations will be unaffected if the form of the new Lagrangian density differs from the form of the old by a divergence term:

$$L'(\varphi_i, \partial_\mu \varphi_i, x^\mu) = L(\varphi_i, \partial_\mu \varphi_i, x^\mu) + \partial_\mu \Theta^\mu. \tag{10}$$

This is the second place in the derivation in which the freedom to pick up a divergence term arises, and the reason is the same: the Euler expression associated with a total derivative vanishes identically.[10] In principle, we could permit the existence of both divergence terms, but in the literature the options seem to be limited to the following two: (a) treat the Lagrangian density as strictly form-invariant, but allow for a non-zero change in the action as indicated in equation (7) above, or (b) treat the action as strictly numerically invariant (so that the surface term in equation (7) vanishes), but allow for the possibility that the Lagrangian density be form-invariant only up to a divergence term, as in equation (10).

In most cases, it is possible to avoid divergence terms altogether. For example, in general relativity the standard electromagnetic action and (Hilbert) gravitational action are each numerically invariant under arbitrary coordinate transformations, with the Lagrangian densities taken to be form-invariant. But this is not possible for the free particle action in classical non-relativistic mechanics when the coordinate transformations include boosts, nor for the 1916 Einstein '$\Gamma\Gamma$' gravitational action[11] – which depends on only first derivatives of the metric – when the coordinate transformations are non-linear. In such cases, it is surely appropriate to

[10] There are some very confusing remarks concerning this point in the literature. Both Hill (1951) and Doughty (1990) restrict the dependence of Θ^μ on the independent and dependent variables to exclude the derivatives of the dependent variables. As they remark, this ensures that the new Lagrangian is of the same order of derivatives of the dependent variable as the original Lagrangian. However, since the resulting Euler–Lagrange equations are independent of any such restriction, it is not clear what the motivation is for imposing this restriction. Furthermore, Hill (1951, p. 257, note 14) indicates that his restriction is not just a sufficient but also a necessary condition for a symmetry transformation. For our purposes, we require only that we have a sufficient condition for a symmetry transformation. (We are already restricting ourselves to theories that can be given a Lagrangian formulation, and to continuous symmetries; the treatment given here is not intended to cover all possible cases of symmetry transformations in physics, but rather to address what follows if certain specified general conditions are satisfied.)

[11] See Dirac (1996, section 26), and Brown and Brading (2002).

choose option (a) above. To choose option (b) strictly means specifying a coordinate system before specifying the form of the Lagrangian (density), something which in practice is never done. In opting for (a) we side with Trautman (1962) and Anderson (1967, p. 91), for example, in opposition to Hill (1952) and Doughty (1990, pp. 190–91) who go for (b).

We are now in a position to connect the sufficient conditions for our transformation to be a symmetry transformation with the *variation* in the action.

Consider the *variation* in the action:

$$\begin{aligned}\delta S &= S[\varphi_i', \partial_\mu \varphi_i', x'^\mu] - S[\varphi_i, \partial_\mu \varphi_i, x^\mu] \\ &= \int_{R'} L(\varphi_i', \partial_\mu \varphi_i', x'^\mu)\, d^4x' - \int_R L(\varphi_i, \partial_\mu \varphi_i, x^\mu)\, d^4x. \end{aligned} \tag{11}$$

First, we add the requirement (9), form-invariance of the Lagrangian density, and we compare this with the change in the action under a general infinitesimal point transformation ΔS (see equation (6)). We see that, under this condition,

$$\Delta S = \delta S. \tag{12}$$

Then, using (8), we conclude that, for an infinitesimal symmetry transformation,

$$\delta S = \int_R \partial_\mu (\Delta \Lambda^\mu)\, d^4x, \tag{13}$$

where the condition (8) is now necessary for the ensuing derivation to proceed.[12]

The next step is to equate (13) with the expression for δS derived above (see equation (4)), giving

$$\begin{aligned}&\int_R \sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) \delta_0 \varphi_i\, d^4x \\ &\quad = \int_R \sum_i \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi_i)} \delta_0 \varphi_i + L\delta x^\mu - \Delta \Lambda^\mu \right) d^4x, \end{aligned} \tag{14}$$

where the left-hand side is the interior contribution and on the right-hand side we have collected all the boundary contributions. Since the region of integration is arbitrary, we therefore arrive at the following solution to the Noether variational problem for symmetries in physics.

'Noether general expression'

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) \delta_0 \varphi_i = -\sum_i \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \varphi_i)} \delta_0 \varphi_i + L\delta x^\mu - \Delta \Lambda^\mu \right) \tag{15}$$

[12] This allows us to prove the most familiar generalization of Noether's own results, as given in what follows. Further generalization is possible by relaxing the condition given in equation (7), leading to more complicated and less familiar results; see below for further discussion of this point, and an illustration from electromagnetism.

This differs from the general solution to Noether's original variational problem (5) by the additional term in Λ^μ appearing on the right-hand side. The connection to symmetries in physics has been made by imposing condition (7) on the action and requiring form-invariance of the Lagrangian density – notice that while these conditions are *sufficient* for the explicit form-invariance of the Euler–Lagrange equations, they are *necessary* for the derivation of the 'Noether general expression' derived here, and in what follows we refer to the class of symmetries satisfying these conditions as 'Noether symmetries'. However, this excludes familiar cases such as the gauge symmetry of Maxwell electromagnetism with a background current, the Lagrangian for which is:

$$L_{EM} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu, \tag{16}$$

where $F^{\mu\nu}$ is defined in terms of A^μ, A^μ being the electromagnetic 4-potential, and J^μ is the 4-current. The Euler–Lagrange equations for the electromagnetic potential are invariant under a gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu \theta, \tag{17}$$

where θ is an arbitrary function of the coordinates. Nevertheless, the variation in the action arising from a gauge transformation is:

$$\delta S = -\int J^\mu \partial_\mu \theta \, d^4x, \tag{18}$$

which does not satisfy condition (7), and hence the 'Noether general expression' (15), derived via (13), is not applicable.[13] Generalized results that include such cases may be derived by considering the *necessary* and sufficient conditions for the explicit form-invariance of the Euler–Lagrange equations.[14] However, this further generalization introduces additional complications, and leads to results that differ from those usually associated with the title 'Noether's theorems'. In our presentation here we give the more familiar results which, in fact, cover almost all cases of interest in physics whilst also displaying clearly the features of Noether's theorems that are of central conceptual interest.

Three theorems are derivable from the 'Noether general expression', two of which are generalized versions of those presented by Noether in her 1918 paper and the third of which derives from Noether's discussions with Klein and his presentation in the context of general relativity in Klein (1918).

[13] For further discussion of this example, in conjunction with Noether's first theorem, see Brading (2002, appendix).

[14] See K. Brading and H. R. Brown, 'Noether's theorems, gauge symmetries and general covariance', unpublished manuscript.

4 Noether's first theorem: global symmetries and conservation laws

The first theorem is the most famous of the three theorems. This theorem applies to 'global' symmetries, where by 'global' we mean, in this context, symmetries depending on constant parameters. Examples of such symmetries are the spacetime symmetries of spatial translations and rotations, temporal translations, and boosts, familiar in Galilean or Lorentzian spacetimes. In sloganized form, Noether's first theorem says that for every continuous global symmetry there exists a conservation law. In fact, the specialization to global symmetries allows us to derive the following result from the Noether general expression (15).

Noether's first theorem

If a continuous group of transformations depending smoothly on ρ constant parameters ω_k ($k = 1, 2, \ldots, \rho$) is a Noether symmetry group of the Euler–Lagrange equations associated with $L(\varphi_i, \partial_\mu\varphi_i, x^\mu)$, then the following ρ relations are satisfied, one for every parameter on which the symmetry group depends:

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \right) \frac{\partial(\delta_0 \varphi_i)}{\partial(\Delta\omega_k)} = \partial_\mu j^\mu_{k(Noether)}, \tag{19}$$

where $\Delta\omega_k$ indicates that we are taking infinitesimal symmetry transformations,

$$\delta_0 \varphi_i = \frac{\partial(\delta_0 \varphi_i)}{\partial(\Delta\omega_k)} \Delta\omega_k, \tag{20}$$

and where $j^\mu_{k(Noether)}$ is the Noether current associated with the kth parameter,

$$j^\mu_{k(Noether)} := -\sum_i \left\{ \frac{\partial L}{\partial(\partial_\mu \varphi_i)} \frac{\partial(\delta_0 \varphi_i)}{\partial(\Delta\omega_k)} + L \frac{\partial(\delta x^\mu)}{\partial(\Delta\omega_k)} - \frac{\partial(\Delta\Lambda^\mu)}{\partial(\Delta\omega_k)} \right\}. \tag{21}$$

Then, if the left-hand side of (19) vanishes we get:

$$\partial_\mu j^\mu_{k(Noether)} = 0, \tag{22}$$

which is a continuity equation for the Noether current. Integrating over an entire space-like surface we obtain, subject to suitable boundary conditions, conservation of the associated Noether charge.[15]

The most familiar applications of Noether's first theorem are in classical particle mechanics, where we link spatial translation, spatial rotation, time translation, and boost symmetries to conservation of linear momentum, angular momentum, energy, and centre-of-mass motion respectively.[16] Perhaps more puzzling is the

[15] For a more detailed discussion of the derivation of Noether's first theorem, see K. Brading and H. R. Brown (op. cit.).

[16] For details see, for example, Landau and Lifshitz (1976, chapter 2) and Doughty (1990, chapter 9).

connection (in relativistic field theory for a complex scalar field coupled to an electromagnetic field)[17] between global gauge symmetry and conservation of electric charge. On one possible account, a state and its gauge-transformed counterpart are mere redescriptions of the same physical situation,[18] from which it might seem that gauge freedom is nothing more than a mathematical freedom of description with no physical significance. And yet conservation of electric charge is surely a physically significant result. At first sight it might seem that Noether's first theorem gives us a very strong result, allowing us to derive a physically significant conservation law from a 'merely mathematical' freedom in how we choose to describe a physical situation. Can it be the case that we are getting physically significant results from 'mere' mathematical inputs?

In fact, there is no mystery here, but the reasons why this puzzle dissolves are informative. First, we need to re-emphasize the point that in order to make the connection between a certain symmetry and an associated conservation law we must ensure that the left-hand side of (19) vanishes, and this will involve using dynamically significant information or assumptions, such as the assumption that all the fields appearing in the theory satisfy Euler–Lagrange equations of motion. If we want to search for the underlying basis of the existence of the general connection between symmetries and conservation laws, then the appropriate place to look is at these assumptions and at their nature – the structure of the Euler–Lagrange equations and the conditions under which these equations are satisfied, for example. For any given Noether symmetry and its associated conservation law, the one can be obtained from the other directly from the Euler–Lagrange equations without the use of Noether's first theorem: the power of Noether's first theorem lies in the fact that it gives us a simple and general mathematical recipe for extracting a Noether conserved quantity from a Noether symmetry,[19] and vice versa, *subject to* the left-hand side of (19) vanishing. Thus, when we use Noether's first theorem to connect a symmetry with a conservation law we have to put in the relevant dynamical information. Nevertheless, it might still seem puzzling that a 'mere freedom in description' should be connected to a physically significant feature, even via the equations of motion. We need to think more carefully about the empirical content of the symmetries themselves.

One way of getting our hands on the empirical significance of a symmetry is through 'Galilean ship' type experiments.[20] Here, we take an effectively isolated subsystem of the universe, transform it (in the case of Galileo's ship we go from the ship being at rest to the ship being in uniform motion), and observe that the

[17] For details see for example Ryder (1996, section 3.3), and K. Brading and H. R. Brown (op. cit.)
[18] The interpretation of gauge transformations is discussed in detail in various papers in Part I of this volume (see Earman, Martin, Norton, Nounou, Redhead, and Wallace).
[19] See Kastrup (1987, p. 140), and also the quotation from Klein given by Pais (1987, p. 610).
[20] See Brown and Sypel (1995) and Budden (1997).

two states of the subsystem are empirically indistinguishable except in relation to (parts of) the rest of the universe. Thus, in the case of Galileo's ship, no experiments carried out inside the cabin of the ship, and without reference to anything outside the ship, enable us to tell whether the ship is at rest or moving uniformly. The two states of motion are empirically indistinguishable except by looking out of the porthole. The very fact that we can get on with reading, writing, eating, dancing, playing tennis, and so forth, while on a cruise-liner is evidence of the symmetry between rest and uniform motion. Similarly, the fact that our bodies, TVs, watches, and computers work the same way whether we are in London or New York is evidence of translational symmetry. Given this very direct empirical significance of the global spacetime symmetries, perhaps it doesn't seem mysterious that something as empirically significant as a conservation law can be derived from a symmetry of the dynamical laws (given that those laws are satisfied).

This is, perhaps, a tempting line of thought, drawing on an important feature of global spacetime symmetries – their active interpretation. On an active interpretation, as understood here, we apply the symmetry transformation to an effectively isolated *sub*system of the universe, yielding two *empirically distinct* scenarios[21] across which the internal evolutions of the subsystem are empirically indistinguishable. (The Galilean ship experiment involves the stronger requirement that we must be able to implement the two distinct scenarios in practice.) However, in the case of global gauge symmetry, this approach doesn't work. Gauge transformations have no analogue to the 'Galilean ship',[22] they have no active interpretation.

While it is true that global gauge symmetry does not have the same direct empirical significance, arising from an active interpretation, that global spacetime symmetries have, this does not imply that global gauge symmetry is without empirical content. *The very fact that a global gauge transformation does not lead to empirically distinct predictions is itself non-trivial.* In other words, the freedom in our descriptions is no 'mere' mathematical freedom – it is a consequence of a physically significant structural feature of the theory. The same is true in the case of global spacetime symmetries: the fact that the equations of motion are invariant under translations, for example, is empirically significant. This is independent of whether there is an active interpretation of the symmetry in question. The imposition of a symmetry on a theory places a restriction on the possible form of the equations of motion of that theory, and insofar as this restriction has empirical significance then so too does the symmetry itself. *This* is the proper place to look when analysing the empirical significance of a given Noether symmetry.

Given a Lagrangian theory, Noether's first theorem applies to any isolated system; thus, it applies to an isolated subsystem of the universe, such as Galileo's ship,

[21] By 'empirically distinct' we mean observationally distinct in principle.
[22] See K. Brading and H. R. Brown (in press).

but it also applies to the universe as a whole. Thinking about Noether's first theorem for the case of the universe as a whole makes vivid the point that the empirical significance of the theorem is not connected to active interpretations of Noether symmetries. For the universe as a whole there is no active interpretation of a symmetry transformation. A Noether symmetry, however, is a property of the equations of motion; thus, insofar as these are well defined for any isolated system, *including the universe as a whole*, all results of the form (19), and indeed the move from (19) to (22) – i.e. linking the symmetry to the continuity equation via satisfaction of the equations of motion – are meaningful. Although the active interpretation of global spacetime symmetries is an important feature of these symmetries, it is a red herring when thinking about Noether's theorems.

In sum, it would be a mistake to think that Noether's first theorem, and indeed the Noether results more generally, allow us to pull physically significant conclusions out of 'merely' mathematical hats: symmetries place restrictions on the form of the equations of motion, and thereby derive much – and in some cases all – of any empirical significance they may have from that of the equations of motion themselves. As we will see in the next section, the second and third theorems derivable from the Noether variational problem show that local gauge symmetry is an even stronger restriction than global gauge symmetry on the possible form of the equations of motion.

5 The Boundary theorem and Noether's second theorem

Turning our attention to local symmetries (i.e. symmetries depending on arbitrary functions of space and time), we can derive two theorems. This is because we can separate the interior and boundary contributions to (14) by noticing that we must allow for the possibility that the arbitrary functions vanish on the boundary,[23] and so we require that the interior and boundary contributions vanish.

5.1 The Boundary theorem

On the basis of the vanishing of the boundary contribution we derive the Boundary theorem from the 'Noether general expression' (15).

The Boundary theorem

If a continuous group of transformations depending smoothly on ρ arbitrary functions of time and space $p_k(x)\,(k = 1, 2, \ldots, \rho)$ and their first derivatives is a Noether symmetry group of the Euler–Lagrange equations associated with $L(\varphi_i, \partial_\mu\varphi_i, x^\mu)$,

[23] For a more detailed derivation of Noether's second theorem and the Boundary theorem, see K. Brading and H. R. Brown, 'Noether's theorems, gauge symmetries and general covariance', unpublished manuscript.

then the following three sets of ρ relations are satisfied, one for every parameter on which the symmetry group depends:

$$\sum_i \partial_\mu \left\{ \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki} \right\} = \partial_\mu j^\mu_{k(\text{Noether})} \tag{23}$$

$$\sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki} = j^\mu_{k(\text{Noether})} - \sum_i \left\{ \partial_\nu \left(\frac{\partial L}{\partial(\partial_\nu \varphi_i)} b^\mu_{ki} + \frac{\partial(\Delta\Lambda^\nu)}{\partial(\partial_\nu \Delta p_k)} \right) \right\} \tag{24}$$

$$\sum_i \left\{ \left(\frac{\partial L}{\partial(\partial_\mu \varphi_i)} b^\nu_{ki} + \frac{\partial(\Delta\Lambda^\nu)}{\partial(\partial_\mu \Delta p_k)} \right) + \left(\frac{\partial L}{\partial(\partial_\nu \varphi_i)} b^\mu_{ki} + \frac{\partial(\Delta\Lambda^\mu)}{\partial(\partial_\nu \Delta p_k)} \right) \right\} = 0 \tag{25}$$

where the infinitesimal transformation $\delta_0\varphi_i$ is given by

$$\delta_0\varphi_i = \sum_k \left\{ a_{ki}(\varphi_i, \partial_\mu\varphi_i, x)\Delta p_k(x) + b^\nu_{ki}(\varphi_i, \partial_\mu\varphi_i, x)\partial_\nu \Delta p_k(x) \right\}, \tag{26}$$

where Δp_k indicates that we are considering infinitesimal transformations, and $j^\mu_{k(\text{Noether})}$ is the Noether current once again, see (21) above, in this case that associated with the kth arbitrary function.

These three identities, along with that of Noether's second theorem (see section 5.2, below), are not independent of one another, but we present all four here since this makes it easier to see their origin and their connection to the related results found in the literature.[24]

The significance of the Boundary theorem is best seen by means of examples, as demonstrated in section 5.3, below. One point is worth making at this stage, however, since it will be needed in the following section in the discussion of Noether's second theorem.[25] The roots of this point are historical, going back to Klein's analysis of Hilbert's energy conservation theorem (see Klein, 1917), and his correspondence with Einstein in March 1918 on the status of energy conservation in general relativity.[26] Indeed, the reasoning leading to the Boundary theorem was first given by Klein (1918) in the context of general relativity, work done in close co-operation with Noether.

Re-arranging the first identity of the Boundary theorem, equation (23), we get:

$$\partial_\mu \left\{ j^\mu_{k(\text{Noether})} - \sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)} \right) b^\mu_{ki} \right\} = 0. \tag{27}$$

[24] Related results can be found in: Barbashov and Nesterenko (1983), de Wet (1947), Govearts (1991), Heller (1951), Julia and Silva, 'Currents and superpotentials in classical gauge invariant theories. I: Local results with applications to perfect fluids and general relativity', E-print gr-qc/9804029 v2 (1998), Klein (1918), Utiyama (1956; 1959), and Weyl (1919).

[25] This material was first presented at the Sixth International Conference on the History of General Relativity.

[26] Einstein (1998).

Hence, defining

$$\Theta^{\mu}_{k} := j^{\mu}_{k(Noether)} - \sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)}\right) b^{\mu}_{ki}, \tag{28}$$

we have that

$$\partial_\mu \Theta^{\mu}_{k} = 0 \tag{29}$$

holds identically. From this, we infer the existence of the so-called 'superpotentials' $U^{\mu\nu}_{k}$, such that

$$\Theta^{\mu}_{k} = \partial_\nu U^{\mu\nu}_{k}, \tag{30}$$

where

$$\partial_\mu \partial_\nu U^{\mu\nu}_{k} = 0 \tag{31}$$

holds identically. All we have done here is some mathematical manoeuvring, allowing us to re-write the Noether current in the following form:

$$j^{\mu}_{k(Noether)} = \sum_i \left(\frac{\partial L}{\partial \varphi_i} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \varphi_i)}\right) b^{\mu}_{ki} + \partial_\nu U^{\mu\nu}_{k}. \tag{32}$$

In other words, the Noether current can be expressed as consisting of a term which vanishes on-shell (i.e. when the Euler–Lagrange equations are satisfied), and a term whose divergence vanishes identically.

Now consider the conservation law

$$\partial_\mu j^{\mu}_{k(Noether)} = 0. \tag{33}$$

Given that the Noether current can be re-written in the form (32), we see that (33) can be understood as the vanishing of the divergence of two contributions. The first contribution vanishes on-shell without any need to take the divergence; the divergence of the second contribution vanishes identically. We can therefore raise a query over what physical significance the vanishing of the divergence of the Noether current can possibly have. This is the basis of the concerns that Klein raised over the status of Einstein's conservation law. At least a part of Einstein's response seems to be that (33) holds only when the field equations are satisfied, and that we are therefore making use of physically significant information in order to move from (32) to (33). This is true, but it doesn't address the full weight of the problem: the term of the Noether current involving the Euler–Lagrange equations vanishes on-shell *without* any need to take the divergence of the Noether current. Taking the divergence plays a role only with respect to the second term, and there the divergence vanishes identically. We are back to the question: wherein lies the physical content in taking the divergence of the Noether current and finding that it vanishes?

Something more can, and should, be said at this point. We have shown that whenever we have a local symmetry the associated Noether current can be re-written in the form (32), such that on-shell

$$j^{\mu}_{k(Noether)} = \partial_\nu U^{\mu\nu}_{k}. \tag{34}$$

Part of the Klein worry is that the associated continuity equation for $j^{\mu}_{k(Noether)}$ lacks physical content because of (31). But notice: while it is true that we can always write an expression of the form (34) on-shell, there remains the question of whether, and if so when, this equation expresses a physically significant relation. So far in doing the re-writing all we have done is mathematics, and only mathematics. The relation (34) gains *physical* significance only when it holds 'not as an identity or definition, but as a field equation postulated to relate two separate systems' (Deser, 1972, p. 1082). Consider, for example, the Maxwell field equations

$$J^{\mu} = \partial_\nu F^{\mu\nu}. \tag{35}$$

These equations are of the form (34), and

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0 \tag{36}$$

holds simply in virtue of the antisymmetry of $F^{\mu\nu}$. Nevertheless, we do not say that conservation of electric charge is a mathematical identity without physical significance. This is because the equations (35) are not a mere mathematical re-expression of the current J^{μ}; they express a physically significant relation between two different types of field: on the left-hand side we have a current, J^{μ}, depending on the matter fields carrying the electric charge, and on the right-hand side we have an expression depending the electromagnetic fields, $F^{\mu\nu}$. Thus, the current conservation law follows, via (35) and (36), and since (35) is physically significant, so is the current conservation law.

Similarly in the case of general relativity, the re-expression of energy–momentum through a relation of the form (34) has physical content because it gives a relation between the behaviour of the metric and the matter fields, it is a field equation with physical content, and hence the conservation law that follows from it (via an identity for the right-hand side) also has physical content. This is perhaps what Einstein was alluding to when he said in a letter to Klein (24 March 1918) that his continuity equation contains 'a part of the content of the field equations'.[27]

5.2 Noether's second theorem

On the basis of the vanishing of the interior contribution we derive Noether's second theorem from the 'Noether general expression' (15).

[27] Einstein (1998, document 492).

Noether's second theorem

If a continuous group of transformations depending smoothly on ρ arbitrary functions of time and space $p_k(x)$ $(k = 1, 2, \ldots, \rho)$ and their first derivatives is a Noether symmetry group of the Euler–Lagrange equations associated with $L(\varphi_i, \partial_\mu\varphi_i, x^\mu)$, then the following ρ relations are satisfied, one for every parameter on which the symmetry group depends:

$$\sum_i \left(\frac{\partial L}{\partial\varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\varphi_i)}\right) a_{ki} = \sum_i \partial_\nu \left\{ b^\nu_{ki} \left(\frac{\partial L}{\partial\varphi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\varphi_i)}\right)\right\} \quad (37)$$

where the infinitesimal transformation $\delta_0\varphi_i$ is given by (26), above.

Thus, we have a dependency between the Euler expressions for the various fields appearing in the theory. Noether's second theorem tells us that in any theory with a local Noether symmetry there is always a *prima facie* case of underdetermination: more unknowns than there are independent equations of motion.[28] This central feature of gauge theories is discussed elsewhere in Part I of this volume; see especially Earman, Norton, and Wallace. The underdetermination means that there are, in general, as many identities involving the fields as there are arbitrary functions involved in defining the local symmetry transformations. In the case of general relativity, for example, inserting the specific form of the Lagrangian density into the second theorem identities associated with general covariance leads to the twice contracted Bianchi identities. These, and their analogues for other symmetries in other theories, are the results most usually associated with Noether's second theorem (sometimes referred to collectively as the 'generalized Bianchi identities').

Noether's main objective in her paper was not this, however. In an exchange with Klein mentioned above (see section 5.1, above, and Klein (1917)), Hilbert conjectured that the difference between generally covariant theories such as general relativity, and earlier theories such as classical mechanics, can be characterized by the differing status of energy conservation: in generally covariant theories the energy conservation law can be re-written, using the Euler–Lagrange equations, such that it holds 'identically'. We have discussed the status of conservation laws in theories with local symmetries in section 5.1, above. What Noether did was to show that this is indeed characteristic of generally covariant theories. Using both her first and second theorems, she showed that the Noether current associated with a *global* symmetry can be re-written in the form (32) *only* when that global symmetry is a special case of a local symmetry. In classical mechanics (for example), the global

[28] For further discussion of exactly how this underdetermination arises, and its connection to the Cauchy problem and to the 'Bianchi identities' (see below), see Anderson (1967, pp. 95–101) and K. Brading and H. R. Brown, 'Noether's theorems, gauge symmetries and general covariance', unpublished manuscript.

space and time symmetry group is *not* a subgroup of a local symmetry group; so, the energy conservation law (associated with global time translations) cannot be re-written in the form (32). Such a conservation law is, according to Hilbert, a 'proper' conservation law, as distinct from the energy conservation laws associated with generally covariant theories.[29] In this way, Noether proved Hilbert's conjecture, and generalized it beyond the case of energy conservation to all continuous global and local symmetry groups.

5.3 Illustrations

What is the general significance of these results? The four identities together place powerful restrictions on the possible form that a theory can take. This is best seen by looking at an example. We choose here to use Weyl's 1918 theory – the original 'gauge' theory – for the purposes of illustration,[30] beginning from Weyl's assumption that, whatever the detailed form of his theory, the following condition must be satisfied (in our notation):[31]

$$\delta S = \int \partial_\mu B^\mu \, dx + \int (W^{\mu\nu}\delta g_{\mu\nu} + \omega^\mu \delta A_\mu)\, dx = 0 \quad (38)$$

where

- $\int \partial_\mu B^\mu \, dx$ represents the boundary contribution to the variation in the action;
- $W^{\mu\nu} = 0$ and $\omega^\mu = 0$ are the gravitational and electromagnetic Euler–Lagrange equations, respectively, whose explicit form is unknown;
- $\delta g_{\mu\nu} = g_{\mu\nu}\Delta\rho$ is an infinitesimal scale transformation involving the metric $g_{\mu\nu}$;
- $\delta A_\mu = \partial_\mu(\Delta\rho)$ is an infinitesimal transformation of the electromagnetic 4-potential A_μ;
- $\rho(x)$ is an arbitrary function parameterizing the local 'gauge' transformations of Weyl's 1918 theory.

From the vanishing of the interior contribution, Weyl derives (Weyl, 1918, p. 32)

$$W^\mu_\mu = \partial_\mu \omega^\mu, \quad (39)$$

which is the result that follows from Noether's second theorem (although Weyl appears to have derived it independently of Noether).[32]

The result (39) expresses a dependence between the Euler expressions associated with the gravitational fields and the electromagnetic fields: this theory has the characteristic underdetermination problem.

[29] See also K. Brading and H. R. Brown (*op. cit.*), and Trautman (1962, p. 179).
[30] See Ryckman and Martin (end of section 2.2), both this volume.
[31] Weyl (1922, pp. 286–9).
[32] See Brading (2002).

For the vanishing of the boundary contribution, Weyl (1922, pp. 286–9) makes explicit use of Klein (1918) to derive the following three sets of equations (in our notation):

$$\begin{aligned} \partial_\mu J^\mu &= \partial_\mu \omega^\mu \\ J^\mu + \partial_\nu F^{\nu\mu} &= \omega^\mu \\ F^{\mu\nu} + F^{\nu\mu} &= 0. \end{aligned} \tag{40}$$

Using the third of these equations, we may re-write the second in the familiar form of the Maxwell Euler expression, $\partial_\nu F^{\mu\nu} - J^\mu$, associated with the Maxwell equations (see Weyl's equation (82)),[33] and current conservation follows when these equations are assumed to be satisfied. These restrictions on the form of Weyl's theory arise from the imposition of local 'gauge' symmetry as specified by the transformations of the metric and vector potential given above, and show how Weyl 'recovered' the structure of Maxwell electromagnetism in his 1918 theory. They follow straightforwardly from the Boundary theorem.

Other illustrations of the power of Noether's second theorem and the Boundary theorem can be found in Utiyama (1959; see also 1956) and the review paper by Barbashov and Nesterenko (1983), along with Brown and Brading (2002), for example.[34] In Brown and Brading (2002) we discuss the case of general covariance and general relativity, where we look at the way in which adding assumptions to 'mere' general covariance gives the results associated with the Noether variational problem increasingly significant bite.[35] We see, for example, the important role of the assumption that the gravitational part of the action be independently invariant (up to a surface term). Also, if the matter part of the Lagrangian density depends on the metric but none of its derivatives, and on the first derivatives only of a vector field, then the gauge structure of those vector fields is dictated. These remarks serve only to hint at the power of the results following from the Noether variational problem. For more details see the papers listed above and the references therein. The results following from the Noether variational problem obviously cannot dictate physical theory to us – they cannot tell us which objects in our theories are physically significant, or which symmetries to impose. Nevertheless, the conjunction of opting to impose a certain symmetry on our physical theory, along with placing restrictions on how that symmetry is to be realized by the mathematical objects appearing in

[33] Weyl uses (39) to arrive at conservation of electric charge in his 1918 theory, $\partial_\mu J^\mu = 0$, via satisfaction of his gravitational field equations (see Brading, 2002). In this derivation he appears to 'help himself' to the Maxwell Euler expression and the antisymmetry of $F^{\mu\nu}$. The second and third equations of (40) are what make this legitimate. We are grateful to Tom Ryckman for bringing the relevant section of Weyl (1922) to our attention.

[34] See also K. Brading and H. R. Brown, 'Noether's theorems, gauge symmetries and general covariance', unpublished manuscript.

[35] This is connected to the 'Kretschmann objection' to Einstein; see Earman (Part I) and Norton (this volume).

the theory, leads to some very strong consequences and restrictions in terms of the possible form that this theory can take.[36]

6 Conclusion

We have explored two aspects of the results relating to Noether's seminal 1918 paper. The first is the relationship between symmetries of the Euler–Lagrange equations and the variational results obtained by Noether and Klein. Their results must be modified to allow for the non-scalarity of the Lagrangian, this being needed for such cases as boosts in Galilean particle mechanics and the arbitrary coordinate freedom of Einstein's 'ΓΓ' Lagrangian.

The second is the significance of the three theorems derivable from the Noether variational problem. Here, we have emphasized the power of the theorems, especially with respect to the structure of gauge theories, while also insisting on the point that the results are mathematics, and their significance in the context of a particular physical theory depends upon interpretational steps that go beyond what the theorems alone can tell us.

Finally, we have also made some remarks concerning the historical origins of the Noether and Klein papers, concerning the issue of whether conservation of energy in general relativity is physically meaningful, and whether this is characteristic of conservation laws connected to local symmetries. Our conclusion is that such conservation laws can indeed be re-written in a characteristic form (this result following mathematically), but that this does not necessarily imply that they are physically meaningless (this issue hinging on further interpretational steps).

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